learning channel
defining *your* destiny.

Mathematics

Learner's Workbook

Grade 11

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Website: www.learn.co.za

First published in 2008

ISBN 978-1-920040-55-2

Edited by Learning Channel
Book design by PH Setting cc
Cover design by Learning Channel
Typesetting by PH Setting cc
Printing by Harry's Printers (Pty) Ltd

Contents

How to use the Learning Channel Mathematics programme for Grade 11

Congratulations and thank you for choosing this Learning Channel Mathematics Grade 11 programme.

This Mathematics programme is comprehensive and covers all the Learning Outcomes, Assessment Standards, knowledge, key concepts and skills for this subject as stated in the National Curriculum Statement – everything you need to make a success of your world. However, it does not replace your teacher or textbook!

This Learning Channel programme is for everyone … you may be using this at home or in your classroom with your teacher and classmates. You may have chosen this programme because you are struggling with Mathematics and as a result you're not achieving the grades you know you deserve. Or you may be using it because it will help you earn the distinction you've set as your goal. Wherever you are and whatever your reason, this programme will give you the head start you need.

The Learning Channel programme consists of three components:

➢ Lessons to watch on DVD;

➢ A learner workbook, with exercises and activities for you to complete; and

➢ If you are connected to the Internet, the Learning Channel website.

Here are some tips on how to make the most of this programme.

Before sitting down to study, make sure you have the following to hand:

- The Learning Channel Mathematics for Grade 11 DVD;
- The Learning Channel Mathematics for Grade 11 Learner's Workbook;
- Pen and paper; and
- Your DVD remote control – if you are watching this on a DVD player.

➢ Insert the Learning Channel Mathematics for Grade 11 DVD disc into your computer or DVD player. Press play.

➢ The subject name and grade will appear, followed by the title of the lesson, the lesson number and the duration of the lesson.

➢ Next, you will be told what lesson to turn to in your workbook.

➢ The Learning Outcomes and Assessment Standards will appear, followed by the lesson overview. This will tell you exactly what you will be expected to do by the end of the lesson.

We suggest that you watch the entire lesson before working in the workbook. While watching the lesson you can stop the DVD when you need to review or refresh what has been said or if you want to take down notes.

While watching the lesson you will also see the PAUSE icon. This alerts you to an activity you can complete in the workbook. If you feel that you are ready to try this concept or skill-related activity press the PAUSE button on your remote control, television or computer screen and complete the activity. Press PLAY once you have completed the activity.

At the end of the lesson you will see a summary of the key concepts covered. If you've been taking notes you can jot these down or find them in your workbook.

➢ All the exercises and activities are designed so that you can complete them on your own. Some activities, however, can also be completed with a partner, in a group or as a class. These opportunities are clearly indicated with icons (see page viii).

➢ Check your answers against the solutions provided at the end of the workbook. Errors may indicate that you have missed or not understood key concepts. Watch the lesson again, refer to any notes you have made and redo the activities you did not master.

The Learning Channel website offers extra features, such as subject glossaries, past exam papers, study tips and the National Curriculum Statement. Visit the website to make use of the extra features.

If you are a teacher using this programme with your class, you will find teacher tips at the end of the Learner's Workbook. These tips will help you use the DVD lesson and convey its content to your learners.

What the icons mean

DVD	DVD – watch the lesson
INDIVIDUAL	Individual work – do this activity on your own
PAIRS	Pair work – do this activity with a partner
GROUPS	Group work – do this activity in a group
CLASS	Class work – do this activity as a class
SELF ASSESSMENT	Self-assessment – assess yourself
BASELINE ASSESSMENT	Baseline assessment – what I know before starting (prior knowledge)
FORMATIVE ASSESSMENT	Formative assessment – how I am progressing
SUMMATIVE ASSESSMENT	Summative assessment – a check of what I know
PROJECT	Project – a project to research and present
COLLECTION OF EVIDENCE	Collection of evidence – include in your portfolio

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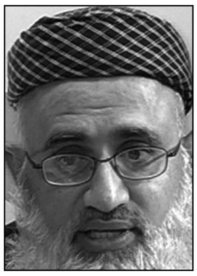
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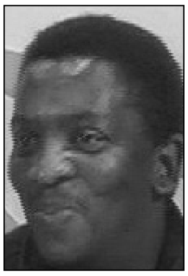
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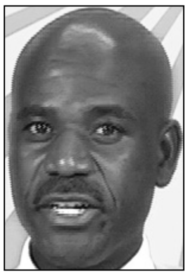
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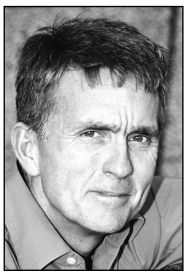
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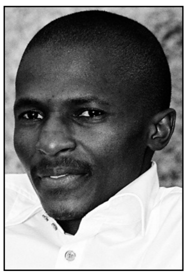
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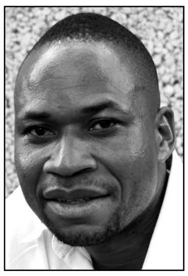
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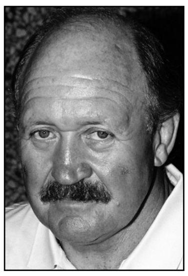
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Other Learning Channel products

Other products in the Learning Channel Grade 11 series:

- Learning Channel English FAL for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Accounting for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Mathematical Literacy for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Life Sciences for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Life Orientation for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Physical Sciences for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel English Home Language for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Afrikaans FAL for Grade 11 DVD lessons and Learner's Workbook
- Learning Channel Business Studies for Grade 11 DVD lessons and Learner's Workbook

The Learning Channel (in conjunction with Liberty Life, Standard Bank, SABC Education and the Department of Education) is one of the world's leading televised learning resources, broadcast on SABC1 on lesson days from 10am to noon.

Its new-look, new-generation content – reflecting South Africa's updated curriculum – has been developed in close collaboration with SABC Education and the Department of Education.

Learning Channel's latest broadcasting endeavours are also supported by a potent mix of delivery platforms – including the Internet, newspapers, hi-tech audio-visual aids, workbooks and SMS – to ensure it maximises its much-needed reach to South Africa's learners.

Learning Channel offers an extensive range of educational material on video or DVD. You can order 15-20 hours of interactive learning with a tutor, accompanied by a workbook to be used in the privacy of your own home or school.

CDs with digitised video lessons are also available.

To order your Learning Channel CDs, DVDs, videos and workbooks, please contact Takalani. E-mail: info@learn.co.za Phone: (011) 214-7100

ALGEBRA

Simplifying algebraic expressions

Lesson 1

Learning Outcomes and Assessment Standards

Assessment standards
(LO2 AS 11.2.4 (b))

Overview

In this lesson you will:

- Revise Grade 10 Algebra.
- Multiply and divide algebraic fractions.
- Add and subtract algebraic fractions.

A. Simplification of fractions

DVD

Rule

- Factorise the denominator
- Factorise the numerator
- Cancel the common fractions

Prior knowledge examples

PAIRS

BASELINE ASSESSMENT

Simplify the following:

1. $\dfrac{x^2 - x - 6}{x^2 - 9}$

$= \dfrac{(x-3)(x+2)}{(x-3)(x+3)}$ Factorise numerator and denominator

$= \dfrac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x+3)}$ Cancel

$= \dfrac{x+2}{x-3}$ This is the simplified form

Pitfall

$$\left.\begin{aligned} &\frac{\cancel{x^2} - x - 6}{\cancel{x^2} - 9} \\ &= \frac{-(x+6)}{-9} \\ &= \frac{x + \cancel{6}^{2}}{\cancel{9}_{3}} \\ &= \frac{x+2}{3} \end{aligned}\right\}$$

One cannot cross cancel over terms!

$\dfrac{x+2}{x-3}$ does not cancel to give $\dfrac{\cancel{x}+2}{\cancel{x}-3} = -\dfrac{2}{3}$ since one cannot cancel over a term.

That is why we factorise the expression first and then cancel common factors.

2. $\dfrac{2(x-2)(x+1) + 3(x-2)}{6x^2 + 13x - 5}$ Take out the common factor $(x - 2)$

$= \dfrac{(x-2)[2(x+1)+3]}{(3x-1)(2x+5)}$ Distribute and simplify in the square brackets

$= \dfrac{(x-2)(2x+5)}{(3x-1)(2x+5)}$

$= \dfrac{(x-2)\cancel{(2x+5)}}{(3x-1)\cancel{(2x+5)}}$ Cancel

$= \dfrac{x-2}{3x-1}$

Example 1

$$\frac{2x^2 + 2ax - 3xy - 3ay}{2x^2 + 3xy - 9y^2}$$

$$= \frac{2x(x + a) - 3y(x + a)}{(2x - 3y)(x + 3y)}$$ (Factorise numerator by grouping in pairs)

$$= \frac{(x + a)\cancel{(2x - 3y)}}{(2x - 3y)\cancel{(x + 3y)}}$$ (Take out the common factor $(x + a)$ in numerator and cancel)

$$= \frac{x + a}{x + 3y}$$

$$\frac{2x^2 + 2ax - 3xy - 3ay}{2x^2 + 3xy - 9y^2}$$

$$= \frac{x(2x - 3y) + a\,(2x - 3y)}{(2x - 3y)(x - 3y)}$$

$$= \frac{\cancel{(2x - 3y)}(x + a)}{\cancel{(2x - 3y)}/(x + 3y)}$$

$$= \frac{x + a}{x + 3y}$$

Example 2 (Additional practise for you)

$$\frac{2x^2 - x - 3}{x^2 + x} \times \frac{3x^2 - x - 4}{6x^2 - 17x + 12} \div \frac{2x + 2}{x^2}$$

$$= \frac{(2x - 3)(x + 1)}{x(x + 1)} \times \frac{(x + 1)(3x - 4)}{(2x - 3)(3x - 4)} \times \boxed{\frac{x^2}{2(x + 1)}}$$ Factorise

$$= \frac{\cancel{(2x - 3)}\cancel{(x + 1)}}{x\cancel{(x + 1)}} \times \frac{\cancel{(x + 1)}\cancel{(3x - 4)}}{\cancel{(2x - 3)}\cancel{(3x - 4)}} \times \frac{x^2}{2\cancel{(x + 1)}}$$ (Cancel)

$$= x$$

To turn the ÷ sign into a × sign, we reciprocate the fraction.

Important facts to remember:

since $(y - x)^2 = (x - y)^2$

and $x + y = y + x$,

so we see that $\frac{x + y}{y + x} = 1$

and $\frac{(x - y)^2}{(y - x)^2} = 1$

but $\frac{x - y}{y - x} = -1$

To change the sign of a fraction, we simply remove the negative, so

$$\frac{p - q}{q - p} = \frac{-(q - p)}{(q - p)} = 1$$

Example 1

$$\frac{x^2 - 6x + 8}{3x + 9} \times \frac{3 + x}{2 - x}$$

$$= \frac{(x - 4)\cancel{(x - 2)}}{3\cancel{(x + 3)}} \times \frac{\cancel{x + 3}}{\cancel{-(x - 2)}}\Bigg\}\ x + 3 = 3 + x \text{ and } 2 - x = -(x - 2)$$

$$= \frac{x - 4}{3}$$

Example 2 (For you to practise)

$$\frac{4 - 9x^2}{6x^2 - x - 2}$$

$$= \frac{(2 - 3x)(2 + 3x)}{(3x - 2)(2x + 1)}$$

$$= \frac{^{(-1)}\cancel{(2 - 3x)}(2 + 3x)}{\cancel{(3x - 2)}(2x + 1)} \quad \left[\text{Since } \frac{(2 - 3x)}{(3x - 2)} = \frac{-\cancel{(3x - 2)}}{\cancel{(3x - 2)}} = -1\right]$$

$$= \frac{(-1)(2 + 3x)}{(2x + 1)}$$

$$= \frac{-2 - 3x}{2x + 1}$$

Worksheet 1

Simplify:

1. $\dfrac{x^2 - x - 6}{18 - 6x}$

2. $\dfrac{9 - 12x + 4x^2}{2x^2 - x - 3} \times \dfrac{4 + x - 3x^2}{6x^2 - 17x + 12}$

3. $\dfrac{(x - 1) - 2(1 - x)^2}{2x^2 + x - 6}$

4. $\dfrac{p^2 - 2p + 15}{6p - 12} \times \dfrac{2 - p}{5 - p}$

5. $\dfrac{xy - 3y + 4x - 12}{3 - x} \times \dfrac{2y}{4y + y^2}$

B. Addition and subtraction of fractions

Rule

- Find the LCM of the denominators (LCD).

Example 1

Write the following as a fraction of the form $\frac{p}{q}$:

$\dfrac{3}{a} + \dfrac{a}{a - 3}$

$= \dfrac{3(a - 3) + a^2}{a(a - 3)}$ (LCD = $a(a - 3)$)

$= \dfrac{3a - 9 + a^2}{a(a - 3)}$

$= \dfrac{a^2 + 3a - 9}{a(a - 3)}$

Example 2

$\dfrac{3}{x + 2} - \dfrac{4}{1 - x} + 2$

$= \dfrac{3}{x + 2} + \dfrac{4}{x - 1} + 2$ Since $1 - x = -(x - 1)$

$= \dfrac{3(x - 1) + 4(x + 2) + 2(x - 1)(x + 2)}{(x + 2)(x - 1)}$ LCD = $(x + 2)\cdot(x + 1)$

$= \dfrac{3x - 3 + 4x + 8 + 2(x^2 + x - 2)}{(x^2 + x - 2)}$

$= \dfrac{7x + 5 + 2x^2 + 2x - 4}{x^2 + x - 2}$

$= \dfrac{2x^2 + 9x + 1}{x^2 + x - 2}$

Example 3

Sometimes we need to factorise denominators first before finding the LCD.

The next example deals with this.

$\dfrac{4}{x^2 - 9} + \dfrac{3}{(x+ 3)^2} - \dfrac{1}{3 - x}$

$= \dfrac{4}{(x - 3)(x + 3)} + \dfrac{3}{(x + 3)^2} + \dfrac{1}{x - 3}$ Remember: $3 - x = -(x - 3)$

$= \dfrac{4(x + 3) + 3(x - 3) + (x + 3)^2}{(x - 3)(x + 3)^2}$ LCD = $(x + 3)^2(x - 3)$

$= \dfrac{4x + 12 + 3x - 9 + x^2 + 6x + 9}{(x - 3)(x + 3)^2}$

$= \dfrac{x^2 + 13x + 12}{(x - 3)(x + 3)^2}$

Now do worksheet 2.

SELF ASSESSMENT

Worksheet 2

Write as a single fraction:

1. $\frac{1}{y-2}+\frac{3}{3-y}-\frac{4}{(3-y)^2}$
2. $\frac{3x}{x+2}-\frac{2}{5-2x}+\frac{4x}{2}$
3. $\frac{6}{9-x^2}-\frac{3}{(x+3)^2}+\frac{1}{x-3}$
4. $\frac{3y+2x}{y+x}-\frac{3}{3y^2+5xy+2x^2}$
5. $x+\frac{1}{x-1}+\frac{2}{2-x}$
6. $\frac{2}{m^2-2m+1}+\frac{3}{1-m}$

DVD

Write as a single fraction:

$$\frac{3}{x-1}+\frac{2}{x}-\frac{4}{3x}$$
$$=\frac{9x+6(x-1)-4(x-1)}{3x(x-1)}$$
$$=\frac{9x+6x-6-4x+4}{3x(x-1)}$$
$$=\frac{11x-2}{3x(x-1)}$$

We keep the denominators when we work with an expression

But

Solve for x: (Multiply each term by the LCD)

$$\frac{3}{x-1}+\frac{2}{x}=\frac{4}{3x}$$

LCD = $3x(x-1)$

So LCD $3x\ (x-1)\neq 0$

When we solve an equation we can get rid of the denominator, provided that we state our restrictions!!

$\therefore\ x\neq 0$ or $x\neq 1$

$\therefore\ \frac{3}{x-1}\times 3x(x-1)+\frac{2}{x}\times 3x(x-1)=\frac{4}{3x}\times 3x(x-1)$

$\therefore\ 9x+6(x-1)=4(x-1)$

$\therefore\ 9x+6x-6=4x-4$

$\therefore\ 11x=2$

$\therefore\ x=\frac{2}{11}$ (which is not 0 nor 1)

INDIVIDUAL

SUMMATIVE ASSESSMENT

Worksheet 3

Practise

1. Simplify: $\frac{2}{2x-3}+\frac{4}{x}$
2. Solve for x: $\frac{2}{2x-3}=\frac{4}{x}$
3. Simplify: $2+\frac{5}{x-1}$
4. Solve for x: $2=\frac{5}{x-1}$
5. Simplify: $\frac{3}{2x}+\frac{4}{3x}-\frac{1}{x+3}$
6. Solve for x: $\frac{3}{2x}+\frac{4}{3x}=\frac{1}{x+3}$
7. Simplify: $\frac{2}{x+1}+\frac{3}{1-x}$
8. Solve for x: $\frac{2}{x+1}+\frac{3}{1-x}=0$

ALGEBRA

Complete the square for expressions

Lesson 2

Learning Outcomes and Assesments Standards

Learning Outcomes 2: Algebra
Assessment Standard
Manipulate algebraic expressions by completing the square

Overview

In this lesson you will:

- Look for patterns in square binomials.
- Learn to complete the square of quadratic binomials expressions.
- Use this concept to find maximum or minimum values.
- Prove that expressions are positive or negative.
- Apply this concept in problem solving.

Lesson

LC DVD

Squares are useful – why?

We know that x^2 is always positive, and when $x = 0$, $x^2 = 0$.

So we write: $x^2 \geq 0$ for all x.

So if we have $(x + 2)^2$, then it will be zero if $x = -2$, but if $x \neq -2$, then it is always positive.

We write $(x + 2)^2 \geq 0$ for all x.

Now what happens when we have $(x + 2)^2 + 4$?

Let's see: $(x + 2)^2 \geq 0$ for all x.

If I add 4 both sides: $(x + 2)^2 + 4 \geq 0 + 4$

So we write: $(x + 2)^2 + 4 \geq 4$ which tells us that it is bigger or equal to 4. So minimum value is 4.

So if I have $(x - 1)^2 - 4$,
then $(x - 1)^2 \geq 0$
so $(x - 1)^2 - 4 \geq -4$
Here the minimum value is -4

- So why are squares useful? They guarantee that the term is positive!
  Also remember: if $(x + 2)^2 \geq 0$, then
  $-(x + 2)^2 \leq 0$
  So $-(x + 2)^2 + 4 \leq 4$
- From $y = ax^2 \pm bx + c$ to $y = a(x \pm p)^2 \pm q$
  If $ax^2 \pm bx + c \quad = a(x \pm p)^2 \pm q$
  Then $ax^2 \pm bx + c \quad = a(x^2 - 2px + p^2) + q$
  $= ax^2 - 2apx + ap^2 + q$
  Divide by a: $x^2 + \frac{b}{a}x + \frac{c}{a} \quad = x^2 - 2px + p^2 + \frac{q}{a}$

Comparing the coefficient:

For x: $\frac{b}{a} = -2p \Rightarrow p = -\frac{b}{2a}$

constant: $\frac{c}{a} = p^2 + \frac{q}{a}$

$c = ap^2 + q \Rightarrow q = c - ap^2$

Look at the following squares and find a pattern.

$(x + 3)^2$
$= x^2 + 6x + (3)^2$

$(x + 4)^2$
$= x^2 + 8x + (4)^2$

$(x - 1)^2$
$= x^2 - 2x + (-1)^2$

$\left(x - \frac{1}{2}\right)^2$
$= x^2 - x + \left(-\frac{1}{2}\right)^2$

What pattern do you see?

The third term of the trinomial is (half of the co-efficient of x term)^2.

Examples

What must be added to the following expressions so that the expressions will result in a perfect square?

1) $x^2 + 6x$
2) $x^2 + 5x$
3) $x^2 - 4x$
4) $x^2 - 3x$
5) $x^2 + \frac{1}{2}x$
6) $x^2 - \frac{2}{3}x$

Solutions

1. $x^2 + 6x + \left(\frac{6}{2}\right)^2$
$= x^2 + 6x + 9$

2. $x^2 + 5x + \left(\frac{5}{2}\right)^2$
$= x^2 + 5x + \frac{25}{4}$

3. $x^2 \left(\frac{-4}{2}\right)^2$
$= x^2 - 4x + 4$

4. $x^2 - 3x + \left(\frac{-3}{2}\right)^2$
$= x^2 - 3x + \frac{9}{4}$

5. $x^2 + \frac{1}{2}x + \left(\frac{\frac{1}{2}}{2}\right)^2$
$= x^2 + \frac{1}{2}x + \left(\frac{1}{2} \times \frac{1}{2}\right)^2$
$= x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2$
$= x^2 + \frac{1}{2}x + \frac{1}{16}$

6. $x^2 - \frac{2}{3}x + \left(\frac{-\frac{2}{3}}{2}\right)^2$
$= x^2 - \frac{2}{3}x + \left(-\frac{1}{3}\right)^2$
$= x^2 - \frac{2}{3}x + \frac{1}{9}$

Now do Activity 1.

Completing the square

Examples

Complete the square for the following expressions:

1. $x^2 - 4x + 10$
$= x^2 - 4x + (-2)^2 - (-2)^2 + 10$
$= x^2 - 4x + 4 - 4 + 10$
$= (x - 2)^2 + 6$

Conclude with $\underbrace{x^2 - 4x + 10}_{\substack{\text{form} \\ ax^2 + bx + c}} = \underbrace{(x - 2)^2 + 6}_{\substack{\text{form} \\ a(x - p)^2 + q}}$

The expression is always positive

The minimum value is 6 when $x = 2$

2. $-x^2 + 3x + 1$

Solution

Take out a factor of -1

$-(x^2 - 3 - 1)$

Add and subtract $\left(-\frac{3}{2}\right)^2$

$= -\left[x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]$

$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]$

$= -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}$

The expression has a maximum value of $\frac{13}{4}$ when $x = \frac{3}{2}$

3. $2x^2 - 3x - 1 = 2\left[x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - \frac{1}{2}\right]$

$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{8}{16}\right]$

$= 2\left(x - \frac{3}{4}\right)^2 - \frac{17}{8}$

$\therefore\ x^2 + 3x + 1 = -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}$

There is an easier way.

$2x^2 - 3x - 1$

$= [16x^2 - 24x - 8]^{\frac{1}{8}}$ $\quad\times 4(2) = 8$ and $\div 8$

$= [16x^2 - 24x + 9 - 9 - 8]^{\frac{1}{8}}$ add $(-3)^2$ and subtract $(-3)^2$

$= [(4x - 3)^2 - 17]^{\frac{1}{8}}$

Practise exercise (work with a friend)

PAIRS

FORMATIVE ASSESSMENT

Complete the square for the following expressions. Attempt the exercise first and then check you answers.

1. $x^2 + 4x + 2$ 2. $2x^2 - 3x + 6$ 3. $-2x^2 + 4x - 5$

Solutions

1. $x^2 + 4x + 2$

$= x^2 + 4x + (2)^2 + 2$ $\left[\text{add and subtract } \left(\frac{4}{2}\right)^2\right]$

$= (x + 2)^2 - 4 + 2$ (form the square)

$= (x + 2)^2 - 2$

The minimum value of the expression is -2.

2. $2x^2 - 3x + 6$

Take out a common factor of 2

$2\left(x^2 - \frac{3}{2}x + 3\right)$

Add $\left(-\frac{3}{4}\right)^2$ and subtract $\left(-\frac{3}{4}\right)^2$ in the brackets.

Factorise the quadratic trinomial

$2[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} + 3]$

Distribute

$2\left(x-\frac{3}{4}\right)^2-\frac{9}{8}+6$

$=2\left(x-\frac{3}{4}\right)^2+\frac{39}{8}$

Conclusions

$2\left(x-\frac{3}{4}\right)^2+\frac{39}{8}$ tells us that the expression is always positive.

The expression has a minimum value of $\frac{39}{8}$ when $x=\frac{3}{4}$.

3. $-2x^2+4x-5$

Solution

$-2\left[x^2-2x+\frac{5}{2}\right]$

$=-2\left[x^2-2x+(-1)^2-1+\frac{5}{2}\right]$

$=-2\left[(x-1)^2+\frac{3}{2}\right]$

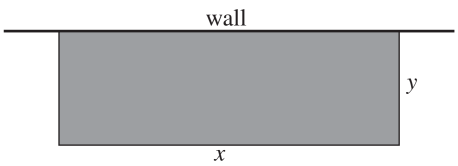
$=-2(x-1)^2-3$

- the expression has a maximum value of –3 when $x=1$.
- The expression is always negative.

Now do Activity 2 numbers 1 to 2

Mathematical modelling

James intends to build a rectangular enclosure for his chickens. He has 100 m of fencing. He intends using a wall for one side of the enclosure.



a) Formulate an expression in x for the area of the enclosure.

b) What will the maximum area of this enclosure be?

c) Which dimensions gives this maximum area?

a) $2y+x=100$

$2y=100-x$

$y=50-\frac{1}{2}x$

$A=x\left(50-\frac{1}{2}x\right)$

$A=-\frac{1}{2}x^2+50x$

b) $A=-\frac{1}{2}[x^2-100x+(-50)^2-2500]$

$A=-\frac{1}{2}(x-50)^2+1250$

Max. area 1 250 m^2

Dimensions

c) $x=50$ $\quad\quad y=25$

Now do Activity 2 no 4.

Activity 1

INDIVIDUAL

FORMATIVE ASSESSMENT

What term must be added to each of the following expressions so that the expression will result in a perfect square.

1. $x^2 + 7x$
2. $x^2 - 6x$
3. $x^2 + \frac{1}{4}x$
4. $x^2 - \frac{2}{5}x$

Activity 2

1. Complete the square of each of the following expressions and write the expression in the form $a(x \pm p)^2 \pm q$

   Say whether the expression has a maximum or minimum value and determine the max or min value.

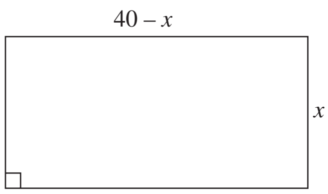
   a) $x^2 + 3x - 7$
   b) $-x^2 - 5x - 1$
   c) $3x^2 + 2x + 7$
   d) $5x^2 + 4x + 3$
   e) $x^2 + px + 3$
   f) $px^2 + qx + r$

2. Prove that the following quadratic expressions are always positive.

   a) $x^2 - x + 5$
   b) $2x^2 - 3x + 8$
   c) $10x^2 + 5x + 2$

3. Prove that the following quadratic expressions are always negative.

   a) $-x^2 + 3x - 5$
   b) $-4x^2 - 2x - 1$

4. 

   i) Find an expression for the area of the above rectangle.
   ii) Find the maximum area of the rectangle.
   iii) What are the dimensions of the rectangle so that the area is a maximum.

NUMBER AND NUMBER RELATIONSHIPS

Exponents

Lesson 3

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and number relationships
Asssessment Standards
We know this when the learner is able to:
- Simplify expressions using the laws of exponents;
- Simplify expressions for rational exponents; and
- Add, subtract, multiply and divide simple surds.

Overview

In this lesson you will:

- Explore laws of exponents covered in Grade 10.
- Explore laws of exponents involving rational exponents and surds.
- Simplify expressions involving exponents and surds.
- Complete activities in the form of worksheets for practice.
- Complete an investigation for your portfolio.

Lesson

LC DVD

Laws of exponents

1. **Product Law**: $a^m \times a^n = a^{m+n}$
2. **Quotient Law:** $\frac{a^m}{a^n} = a^{m-n}$

   From the Quotient Law we get:

   a) Negative exponents

   $\frac{a^3}{a^4} = a^{-1} = \frac{1}{a}$ so

   $4p^{-1}q^2 = \frac{4q^2}{p}$ and

   $3m^{-1} = \frac{3}{m}$ and

   $\frac{6}{x^{-2}} = 6x^2$ and

   $a + b^{-1} = a + \frac{1}{b}$ and

   $(x + 3)^{-2} = \frac{1}{(x+3)^2}$

   and $(2m)^{-2} = \frac{1}{(2m)^2} = \frac{1}{(4m)^2}$

   b) The exponent zero

   $\frac{a^4}{a^4} = a^0 = 1$ so $6^o = 1$

   $3p^0 = 3$ but $(3p)^0 = 1$

   $3 + m^0$ but $(3 + m)^0$

   $= 3 + 1$ $=1$

   $= 4$

   $a + b^0$ but $(a + b)^0$

   $= a + 1$ $= 1$

The Power Law

Example of using the Power Law

$(a^m)^n = a^{mn}$

$(a^mp^n)^q = a^{mq} \cdot p^{nq}$

$\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{nq}}$

Example

$(2m^2n^{-3}p)^{-2}$

$= 2^{-2}\cdot m^{-4}\cdot n^6\cdot p^{-2}$

$= \dfrac{n^6}{4m^4p^2}$

Rational exponents and roots

$\sqrt[n]{m} = m^{\frac{1}{n}} \qquad \sqrt[n]{a^m} = a^{\frac{m}{n}}$

so in general

$\sqrt{p} = p^{\frac{1}{2}} \qquad \sqrt[3]{m} = m^{\frac{1}{3}} \qquad \sqrt[3]{a^2} = a^{\frac{2}{3}} \qquad \sqrt{m^5} = m^{\frac{5}{2}}$

From Rational Exponents and Roots

1.	$4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2$	2.	$\sqrt[3]{8} = 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2$
3.	$(27)^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$	4.	$(8)^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}} = 2^{-2} = \frac{1}{4}$
5.	$\sqrt[4]{16^3} = (16)^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 8$	6.	$\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{2^3}{3^3}\right)^{-\frac{2}{3}} = \frac{2^{-2}}{3^{-2}} = \frac{9}{4}$

Watch the negative sign

1.	$-9^{\frac{1}{2}} = -(32)^{\frac{1}{2}} = -3$	2.	$(-9)^{\frac{1}{2}}$ is invalid. Why?
3.	$-8^{\frac{1}{3}} = -(2^3)^{\frac{1}{3}} = -2$	4.	$(-8)^{\frac{1}{3}} = (-2^3)^{\frac{1}{3}} = -2$ Why?
5.	$(-27)^{\frac{2}{3}} = (-3^3)^{\frac{2}{3}} = (-3)^2 = +9$	6.	$(-9)^{\frac{3}{2}}$ is invalid
7.	$(-27)^{-\frac{2}{3}} = (-3^3)^{-\frac{2}{3}} = (-3)^{-2} = \frac{1}{9}$	8.	$(-16)^{-\frac{3}{2}}$ is invalid

We can't find the square root of a negative (this applies to all even roots).

Example 1

Simplify:

$\dfrac{6^{x+1}\cdot 4^{x-1}\cdot 3^{x-2}}{12^{x+1}\cdot 2^x\cdot 3^{x-1}}$

$= \dfrac{(3^1\cdot 2^1)^{x+1}\cdot (2^2)^{x-1}\cdot 3^{x-2}}{(2^2\cdot 3)^{x+1}\cdot 2^x\cdot 3^{x-1}}$

$= \dfrac{3^{x+1}\cdot 2^{x+1}\cdot 2^{2x-2}\cdot 3^{x-2}}{2^{2x+2}\cdot 3^{x+1}\cdot 2^x\cdot 3^{x-1}}$

$= 3^{x+1+x-2-x-1-x+1}\cdot 2^{x+1+2x-2-2x-2-x}$

$= 3^{-1}\cdot 2^{-3}$

$= \frac{1}{3}\times\frac{1}{8} = \frac{1}{24}$

Note that this expression does not contain any terms!

So we:

(a) prime the bases

(b) collect like bases

Example 2

$\dfrac{16\cdot 8^n}{4^{n+1}\cdot 2^{n+4}}$

$= \dfrac{2^4\cdot (2^3)^n}{(2^2)^{n+1}\cdot 2^{n+4}}$ Prime every base

$= \dfrac{2^4\cdot 2^{3n}}{2^{2n+2}\cdot 2^{n+4}}$ Drop the brackets

$= 2^{4+3n-2n-2-n-4}$ Collect powers

$= 2^{-2}$ Add/subtract powers

$= \dfrac{1}{2^2}$

$= \dfrac{1}{4}$

If no terms present:

(a) prime the bases

(b) collect like bases

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

Simplify.

1. $\left(\dfrac{27a^4}{8a}\right)^{\frac{2}{3}}$

2. $\left(\dfrac{16x^{-3}y^4}{9xy^{-2}}\right)^{-\frac{1}{2}}$

3. $\dfrac{10^{n+3}\cdot 5^{n-1}}{50^{n+2}}$

4. $\dfrac{9^{2k+1}\cdot 3^{-k}\cdot 4^{2-k}}{3^{5k-2}\cdot 6^{3-2k}\cdot 2}$

5. $\dfrac{3^{3n-1}\cdot 9^{-4n}\cdot 27^{n+1}}{3^{-(3n+1)}}$

6. $\dfrac{3^x\cdot 6^{2x-1}\cdot 2^{4x+1}}{12^{3x}}$

DVD

Activity 2

Example of addition and subtraction with exponents.

Rule Break down and factorise

Example:

1. $\dfrac{3\cdot 2^{x-2} - 2^{x-1}}{2^x + 3\cdot 2^{x-3}}$

$= \dfrac{3\cdot 2^x\cdot 2^{-2} - 2^x\cdot 2^{-1}}{2^x + 3\cdot 2^x\cdot 2^{-3}}$ Common factor

$= \dfrac{2^x(3\cdot 2^{-2} - 2^{-1})}{2^x(1 + 3\cdot 2^{-3})}$

$= \dfrac{\cancel{2^x}\left(\frac{3}{4} - \frac{1}{2}\right)}{\cancel{2^x}\left(1 + \frac{3}{8}\right)}$

Multiply each term by LCD = 8

$= \dfrac{6 - 4}{8 + 3}$

$= \dfrac{2}{11}$

Watch out! We have terms, so we

(a) prime the bases

(b) search for a common factor and factorise

(c) simplify by cancelling the common factors

INDIVIDUAL

SUMMATIVE ASSESSMENT

Simplify:

1. $\dfrac{3^x + 3^{x+1}}{3^x - 3^{x+2}}$

2. $\dfrac{2\cdot 2^{x+1} + 8\cdot 2^{x-3}}{4\cdot 2^{x-1} - 16\cdot 2^{x-4}}$

3. $\dfrac{6\cdot 3^{x-1} - 2\cdot 3^{x+1}}{5\cdot 3^x + 3^{x+2}}$

4. $\dfrac{5^x + 5^{x-2}}{2\cdot 5^{x-1} - 3\cdot 5^{x-2}}$

5. $\dfrac{(a - a^{-1})^2\cdot(a-1)^{-1}}{a^{-2} + 2a^{-1} + 1}$

$= \dfrac{\left(a - \frac{1}{a}\right)^2}{(a-1)^1\cdot\left(\frac{1}{a^2} + \frac{2}{a} + 1\right)}$ Remember: $a^{-m} = \dfrac{1}{a^m}$

$= \dfrac{\left(\frac{a^2-1}{a^2}\right)}{(a-1)\left(\frac{a^2+2a+1}{a^2}\right)}$ Add inside brackets with LCD

$= \dfrac{(a-1)(a+1)a^2}{(a-1)(a+1)^2a^2}$ Simplify

$= \dfrac{1}{a+1}$

INVESTIGATION 1

LC PORTFOLIO

Real numbers are divided into rational numbers and irrational numbers.

Rational numbers can be written as $\frac{a}{b}$ where a and b are both integers.

Examples are $\frac{3}{4}$; 4; 1,3; $1\frac{1}{4}$; $2,\dot{1}\dot{4}$.

We know exactly where to put rational numbers on the number line.

Irrational numbers

Cannot be written as $\frac{a}{b}$ where a and b are both integers.

Examples are

$\sqrt{2}$; $\sqrt{5}$; $\sqrt[3]{9}$; π. We know that $\sqrt{2}$ is somewhere between 1 and 2 – closer to 1 than 2, but we are not sure where to put $\sqrt{2}$on the number line.

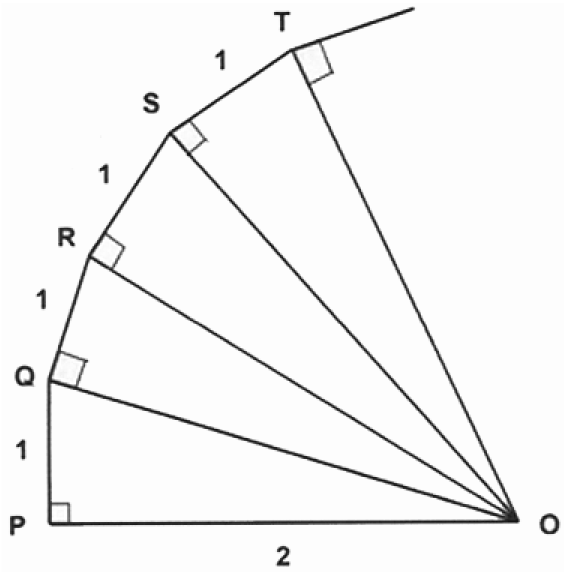
a) Can you use a number line and a construction to put $\sqrt{2}$ accurately on the number line. How about $\sqrt{5}$; $\sqrt{13}$; $\sqrt{20}$; $\sqrt{34}$; $\sqrt{45}$?

b) What common property do the above surds have? Make up some more.

c) Now use what you discovered above to put $\sqrt{3}$ $\sqrt{6}$ $\sqrt{7}$; (in fact, any surd) accurately on the number line by construction. (Approach your teacher if you need help).

INVESTIGATION 2

A sequence of triangles OPQ, OQR, ORS . . . is formed as shown below.



Calculate the lengths of OQ; OR; OS; OT (leave your answer in simplest surd form).

Write in terms of n, the hypotenuse of the nth triangle.

Investigate for different lengths of PO.

Lesson 4

QUADRATIC EQUATIONS WITH FRACTIONAL DENOMINATORS

Learning Outcomes and Assessment Standards

Learning Outcomes 2: Functions and algebra
Assessment Standard
Solve quadratic equations by factorising.

Overview

In this lesson you will:

- Factorise the quadratic trinomial to solve the equation.
- Learn when equations are not defined.
- Use mathematical modelling – use algebra to prove conjectures.

LC DVD

Lesson

If $a{\cdot}b = 0$

Then $a = 0$ or $b = 0$

Note: For this to be true, the product must equal zero. Note that only one of the factors need to be zero, to make the product zero, since zero times anything is zero.

Rules for solving quadratic equations

- If there are denominators , factorise each denominator and find the LCD.
- Write down the LCD and make note of the restrictions. (No denominator can be zero because division by zero is undefined.)
- Multiply each term by the LCD.
- Make the quadratic equation equal to zero.
- Factorise and solve.

Example 1

Solve for x:

$x - 1 = \frac{6}{x}$

LCD $= x$ restriction: $x \neq 0$

$x^2 - x = 6$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

Example 2

Solve for x:

$\frac{2x}{1-x} - \frac{x}{x-1} = \frac{2}{x+1}$ restrictions: $x \neq 1$; $x \neq -1$

$\therefore -2x(x + 1) - x(x + 1) = 2(x - 1)$

$\therefore -2x^2 - 2x - x^2 - x = 2x - 2$

$\therefore 3x^2 + 5x - 2 = 0$

$\therefore (3x - 1)(x + 2) = 0$

$x = \frac{1}{3}$ or $x = -2$

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FORMATIVE ASSESSMENT

INDIVIDUAL

Example 3 (for you to practise)

$$\frac{x}{x+1} - \frac{2x}{1-x} = \frac{x^2+3}{x^2-1} + \frac{9}{4}$$

$$\frac{x}{x+4} + \frac{2x}{x-1} = \frac{x^2+3}{(x-1)(x+1)} + \frac{9}{4} \qquad \text{(change sign)}$$

Restr: $x \neq \pm 1$ LCD $4(x-1)(x+1)$

$4x(x-1) + 2x \cdot 4(x+1) = (x^2+3)4 + 9(x^2-1)$ (get rid of fraction)

$4x^2 - 4x + 8x^2 + 8x = 4x^2 + 12 + 9x^2 - 9$

$\therefore x^2 - 4x + 3 = 0$ (standard form)

$\therefore (x-1)(x-3) = 0$

$x \underbrace{= 1}_{}$ or $x = 3$

na
since $x \neq 1$

The denominator may not be zero

Special results

DVD

If $5x + 2 = 5x + 3$

$0 = 5$ or $2 = 3$

This is not possible since 2 can never equal 3.

So there is no solution

But if $5x - 2 = 5x - 2$, we have many solutions. No matter what we make x, the equation holds true. This is referred to as an indetermined equation.

$0 = 0$. This is always true.

So $x \in \mathbb{R}$

In other words, x can be any real number

Example 4

$$2 - \frac{2x-3}{x-2} = \frac{1}{2-x}$$

$$\therefore 2 - \frac{2x-3}{x-2} = \frac{-1}{x-2}$$

Restr: $x \neq 2$ LCD $(x-2)$

$\therefore 2(x-2) - 2x + 3 = -1$

$\therefore 2x - 4 - 2x + 4 = 0$

$0 = 0$

and $0 = 0$ is always true.

So an infinate number of values for x will satisfy this equation.

So $x \in \mathbb{R} \setminus \{2\}$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

Solve for x:

1. $x(x + 4) = 21$
2. $x(x - 1) = 4(3x - 10)$
3. $(2x - 5)(3x + 2) = 2(3x - 1)$
4. $\frac{x+2}{x+1} - \frac{3}{x-2} = \frac{1}{x+1}$
5. $\frac{30}{x-2} - \frac{1}{2} = \frac{30}{x}$
6. $\frac{4x}{3x+12} - \frac{1}{2} = \frac{1}{2x-2}$
7. $\frac{21}{8(x-6)} - \frac{5}{8(x+2)} + 1 = 0$
8. $\frac{x+1}{x} - \frac{5x}{3x+3} = \frac{2}{3}$
9. $\frac{x+1}{x^2-4} + \frac{1-x}{x+2} = \frac{2}{5(x-2)}$
10. $\frac{2+\frac{1}{x}}{2} = \frac{3-\frac{1}{x}}{3+\frac{1}{x}}$

    Solve for x if

    a) x is an integer
    b) is a rational number
    c) x is an irrational number
    d) x is a real number

DVD

PAIRS

FORMATIVE ASSESSMENT

Mathematical modelling

The use of algebra to prove conjectures

Example 1 (for you to practise)

Look at any five consecutive numbers

Let's take 1 2 3 4 5

Explore the difference between the product of the last two and the product of the first two (4)(5) – (1)(2) = 18

Investigate many more and what do you notice each time?

Now you will need to make a conjecture – this means you must state in words exactly what you have observed.

"In a sequence of five consecutive numbers, the difference between the product of the largest two and the smallest two will always be six times the middle number."

Now we are going to use algebra to prove this conjecture.

A conjecture is not a tested and proved fact, it is merely an inferred conclusion based on an observation.

Proof

Let the five consecutive numbers be $(x - 2)$; $(x - 1)$; x; $x + 1$; $x + 2$

$(x + 2)(x + 1) - (x - 2)(x - 1)$

$= x^2 + 3x + 2 - (x^2 - 3x + 2)$

$= x^2 + 3x + 2 - x^2 + 3x - 2$

$= 6x$

Some useful results you should know

$2n$ will always be an even number so $2n \pm 1$ will always be an odd number.

Activity 2

PORTFOLIO

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. Prove that the sum of two odd numbers is always even.
2. Prove that the difference between any two consecutive square numbers is an odd number.
3. a) Show that the difference between the 7^{th} square number and the 4^{th} square number is a multiple of 3.

   b) Show that the difference between the 10^{th} square number and the 6^{th} square number is a multiple of 4.

   c) Show that the difference between the 12^{th} square number and the 7^{th} square umber is a multiple of 5.

   d) Make a conjecture.

   e) Prove your conjecture.

Lesson 5

QUADRATIC EQUATIONS

Involving Surds

Learning Outcomes and Assesments Standards

Learning Outcomes 2: Functions and algebra
Assessment Standard
Solutions of a quadratic equation with a single surd.

Overview

In this lesson you will:

- Examine when a surd expression is real and solve the surd equation.
- Examine when a surd equation is defined.

LC DVD

Lesson

Quadratic equations

If $x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

Remember:
$\sqrt{4} = 2$ (positive) and $-\sqrt{4} = -2$ (negative) and $\sqrt{-4}$ is non-real

Why are roots (surds) so tricky?

Can we determine $\sqrt{-4}$? The answer is yes, but if we allow non $\mathbb{R}$ numbers. At school we work with Real ($\mathbb{R}$) numbers only, so $\sqrt{-4}$ is not possible!

So when we see a root sign, we must make sure that what is underneath that root sign, is positive in value.

So if $y = \sqrt{x}$, then $x \geq 0$

Note that we have $y = +\sqrt{x}$, so $y \geq 0$
if $x \geq 0$

If we have $y = -\sqrt{x}$, then $y \leq 0$
if $x \geq 0$

In both cases, if x is not positive, we cannot determine y.

This forms the conceptual basis that builds sufficiently for flexible application!

So for $y = \sqrt{x + 1}$; $x \geq -1$ and $y \geq 0$
$y = 2\sqrt{x + 1} - 4$; $x \geq -1$ and $y \geq -4$

Method of solving a surd equation

- Isolate the surds
- State restrictions on x
- Square both sides
- Get into standard form
- Solve for x
- Check your solution against the restriction.

Examples

1. Solve for x.

$\sqrt{x+6} - x = 4$ Isolate the surd

$\sqrt{x+6} = 4 + x$

Remember:

$(a+b)^2 = a^2 + 2ab + b^2$

Now state restrictions on x.

$x + 6 \geq 0 \quad \therefore x \geq -6$

$4 + x \geq 0 \quad \therefore x \geq -4$

Thus $x \geq -4$

$x + 6 = (4 + x)^2$ Square both sides, not each term

$x + 6 = 16 + 8x + x^2$

$0 = x^2 + 7x + 10$ Get into standard form

$0 = (x + 5)(x + 2)$ Factorise

$x = -5$ or $x = -2$ Solve

But $4 + x \geq 0 \therefore x \geq -4$ Check answer with restriction. Omit the solution
$\therefore x \neq -5$ that does not satisfy the restriction

$\therefore x = -2$

2. $\sqrt{2x+1} + 7 = x$

$\therefore \sqrt{2x+1} = x - 7$ (Isolate the surd)

Restriction: $2x + 1 \geq 0$ and $x - 7 \geq 0$

$\underbrace{x \geq -\frac{1}{2} \qquad x \geq 7}_{\therefore x \geq 7}$

Now: $2x + 1 = (x - 7)^2$ (square both sides)

$\therefore 2x + 1 = x^2 - 14x + 49$

$\therefore x^2 - 16x + 48 = 0$

$\therefore (x - 12)(x - 4) = 0$

$\therefore x = 12$ or $x = 4$

But $x \geq 7$

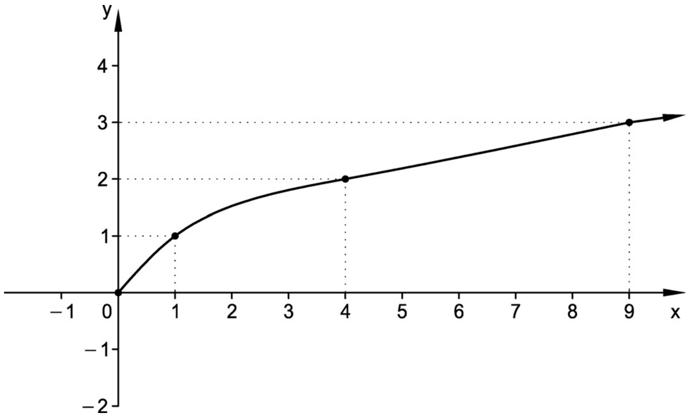
Thus $x = 12$ only.

Linking to graphs

Remember that $x \geq 0$, so we only use positive values of x in our table.

1. Let's sketch a graph $y = \sqrt{x}$.

x	0	1	4	9
y	0	1	2	3



Domain (D_f) = $\{\frac{x}{x} \geq 0\}$

Range (R_f) = $\{\frac{y}{y} \geq 0\}$

Alternatively, use interval notation:

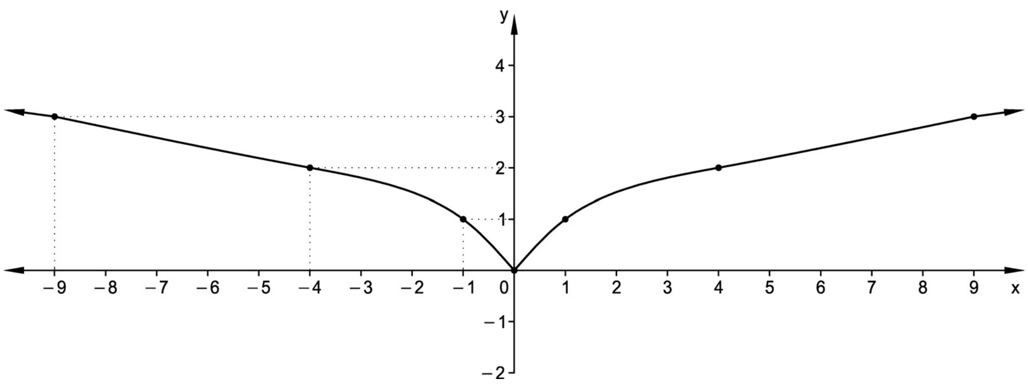
$D_f = [0; \infty)$

$R_f = [0; \infty)$

2. What happens if you reflect the graph in the y-axis?

x changes sign.

New graph:



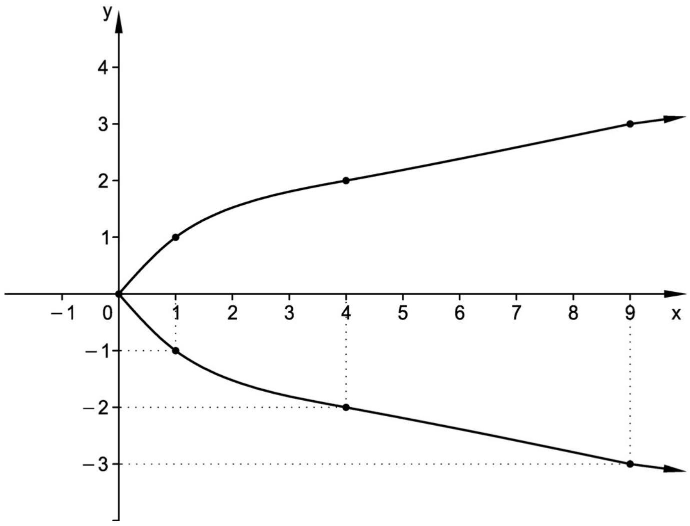
For $y = \sqrt{-x}$

Domain = $\{\frac{x}{x} \leq 0\} x \in \mathbb{R}$

Range = $\{\frac{y}{y} \geq 0\}\ y \in \mathbb{R}$

3. What happens is you reflect the graph in the x–axis?

y changes sign



For $y = -\sqrt{x}$

Domain = $\{\frac{x}{x} \geq 0\}\ x \in \mathbb{R}$

Range = $\{\frac{y}{y} \leq 0\}\ y \in \mathbb{R}$

Alternatively, use interval notation:

Domain = $\{0;\infty\}$

Range = $\{-\infty;0\}$

4. Sketch graph of $f\,(x) = \sqrt{x-2}$

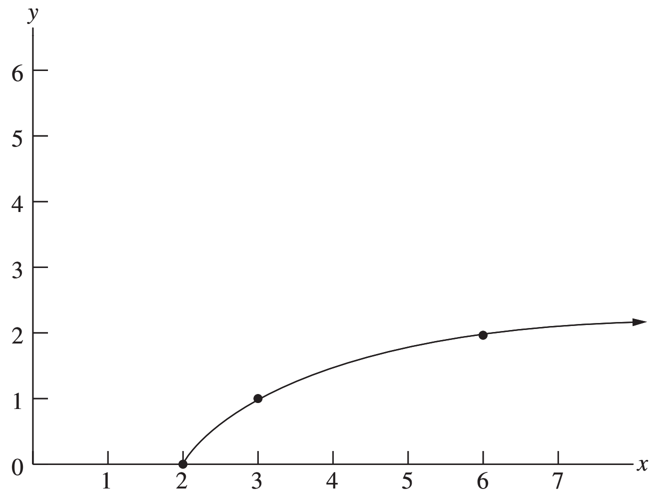
Restriction: $x - 2 \geq 0$

$x \geq 2$

x	2	3	6	11
y	0	1	2	3

D_f: $x \geq 2$; $x \in \mathbb{R}$

R_f: $y \geq 0$; $y \in \mathbb{R}$



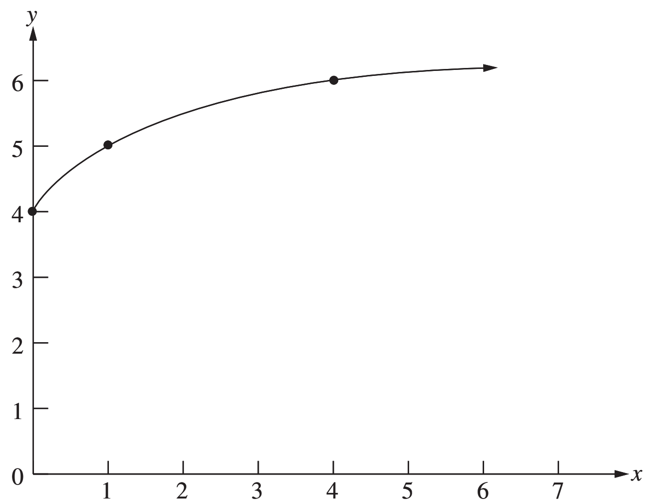
5. Sketch graph of $f(x) = \sqrt{x + 4}$

Restriction: $x \geq 0$

x	0	1	4
y	4	5	6

D_f: $x \geq 0$; $x \in \mathbb{R}$

R_f: $y \geq 4$; $y \in \mathbb{R}$



Now do Activity 2.

In summary:

If we reflect in the y-axis; $x \leftrightarrow -x$

If we reflect in the x-axis; $y \leftrightarrow -y$

If we reflect in the line $y = x$; $x \leftrightarrow y$

LC INDIVIDUAL

LC SUMMATIVE ASSESSMENT

Activity 1

1. Solve for x:
   - (a) $2 = x - \sqrt{2x - 1}$
     $\sqrt{x + 6} - x = 4$
     $x - 2\sqrt{x - 3} = 3$
   - (b) $\sqrt{11 - x} - x = 1$
   - (c) $\sqrt{5x - 1} + 1 - 2x = 0$
   - (d) $x + \sqrt{x - 2} = 4$
   - (e) $\sqrt{10 - 3x} = x - 2$
   - (f) $x\sqrt{x - 2} - 5x - 3\sqrt{x - 2} + 15 = 0$
2. If given $\sqrt{8 - 2x} = \frac{x}{2} + 1$
   - (a) Show that $-2 \leq x \leq 4$
   - (b) Solve for x
   - (c) Without any further calculation, solve the equation
     $-\sqrt{8 - 2x} = \frac{x}{2} + 1$
3. Solve for x. $\sqrt{x - 5} - 4 = \frac{12}{\sqrt{x - 5}}$
4. Solve for x: $\sqrt{\frac{12x + 8}{x - 1}} = 3 - \sqrt{\frac{x - 1}{3x + 2}}$

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

PORTFOLIO

(Investigation)

1. Sketch the graph of $y = \sqrt{-x}$ and write down the domain and range.
2. If $m = \sqrt{n}$ give all possible real values of m and all possible real values of n.
3. If a = $-\sqrt{-c}$ give all possible real values of a and all possible real values of c.

Lesson 6

QUADRATIC EQUATIONS

Completing the square

Learning Outcomes and Assesments Standards

Learning Outcomes 2: Functions and algebra
Assessment Standard
Solutions of a quadratic equation by completing the square and the derivation of the quadratic formula.

Overview

In this lesson you will:

- Learn an algorithm to enable you to write a quadratic equation in the completed square format.
- Solve the equation.
- Decide when there are solutions and when there are no solutions.
- Use this technique to develop a formula to solve quadratic equations.

Lesson

LC DVD

Solving equations by completing the square and developing a formula to solve quadratic equations.

Look at the following example:

Perfect square equation

$(x - 3)^2 = 8$

$x - 3 = \pm\sqrt{8}$

$x = 3 \pm \sqrt{8}$

It is sometimes asked to give the answer in simplest surd form:

Then $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$

So $x = 3 \pm 2\sqrt{2}$

The non-perfect square equation

$x^2 - 3x - 5 = 0$

$x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 - 5 = 0$

We add and subtract the square of half the number before x term and we get

$x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \frac{9}{4} - 5 = 0$

Factorise the trinomial

$\left(x - \frac{3}{2}\right)^2 = \frac{9}{4} + 5$

Simplify

$\left(x - \frac{3}{2}\right)^2 = \frac{9 + 20}{4}$

$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}}$

$x = \frac{3 \pm \sqrt{29}}{2}$ Note that this means that x is irrational since $\sqrt{29}$ is an irrational number

The number before the x term is called the coefficient of x

Let's do another one

Solve by completing the square

$2x^2 + x - 8 = 0$

Step 1: Divide by 2

$x^2 + \frac{1}{2}x - 4 = 0$

Why can we divide by 2?

Because we are working with an equation (and not an expression) of which the one side is zero, so we can lose the two in the front.

Step 2: Space out

$x^2 + \frac{1}{2}x \qquad -4 = 0$

Step 3: Add and subtract the square of half the number before *x*

$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 4 = 0$

If you are afraid of working with fractions, here is an easier method which uses the square completion to solve an equation.

For $ax^2 + bx + c = 0$
(1) Multiply any term with $4a$
(2) Add b^2 to each side
(3) You solve.

Let's see:

- $x^2 - 3x - 5 = 0$ $\quad a = 1$ so $4a = 4$ and $b^2 = (-3)^2 = 9$

$4x^2 - 12x + 9 - 20 = 0 + 9$ [We multiplied each term with 4 and added 9 to each side]

$\therefore (2x - 3)^2 = 29$

$\therefore 2x - 3 = \pm\sqrt{29}$ No fractions

$\therefore 2x = 3 \pm\sqrt{29}$

$x = \frac{3 \pm\sqrt{29}}{2}$

- $2x^2 + x - 8 = 0$ $\quad a = 2$ so $4a = 4(2) = 8$ and $b^2 = (1)^2 = 1$

$\therefore 8(2x^2 + x - 8) + 1 = 0 + 1$

$\therefore 16x^2 + 8x - 64 + 1 = 1$

$\therefore 16x^2 + 8x + 1 = 65$

$\therefore (4x + 1)^2 = 65$

$4x + 1 = \pm\sqrt{65}$ No fractions

$4x = -1 \pm\sqrt{65}$

$x = \frac{-1 \pm\sqrt{65}}{4}$

Step 4: Leave the trinomial on its own

$x^2 + \frac{1}{2}x = \left(\frac{1}{4}\right)^2 = \frac{1}{16} + 4$

Step 5: Factorise the trinomial and simplify the right hand side

$\left(x + \frac{1}{4}\right)^2 = \frac{1 + 64}{16}$

Solve

$x + \frac{1}{4} = \pm\frac{\sqrt{65}}{4}$

$x = -\frac{1}{4} \pm \frac{\sqrt{65}}{4}$

Careful of negatives

Solve by completing the square

Method 1

$-3x^2 - 6x + 1 = 0$

$x^2 + 2x - \frac{1}{3} = 0$

$x^2 + 2x + (1)^2 - (1)^2 - \frac{1}{3} = 0$

$x^2 + 2x + (1)^2 = 1 + \frac{1}{3}$

$(x + 1)^2 = 1\frac{1}{3}$

$x + 1 = \pm\sqrt{\frac{4}{3}}$

$x = -1 \pm \frac{2}{\sqrt{3}}$

$x = \frac{-\sqrt{3} \pm 2}{\sqrt{3}}$

Method 2

$-3x^2 - 6x + 1 = 0 \qquad 4(-3) = -12$
$\qquad b^2 = 36$

$\therefore\ -12(-3x^2 - 6x + 1) + 36 = 36$

$\therefore\ 36x^2 + 72x - 12 + 36 = 36$

$\therefore\ (36x^2 + 72x + 36) = 36 + 12$

$(6x + 6)^2 = 48$

$\therefore\ 6x + 6 = \pm\sqrt{48}$

$6x + 6 = \pm 4\sqrt{3}$

$x = \frac{+2(-3 \pm 2\sqrt{3})}{6}$

$x = \frac{-3 \pm 2\sqrt{3})}{3}$

Now do Activity 1 Nos 1–7.

General form

Solve by completing the square

$ax^2 + bx + c = 0$

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0 \qquad \times 4a$

$4a^2x^2 + 4abx + 4ac = 0 \qquad \text{add } b^2$

$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$

$\therefore\ (2ax + b)^2 = b^2 - 4ac$

$2ax + b = \pm\sqrt{b^2 - 4ac}$

$2ax = -b \pm \sqrt{b^2 - 4ac}$

$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Formula to solve quadratic equations

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

so

Solve: $3x^2 - 4x - 5 = 0$

$a = 3 \quad b = -4 \; c = -5$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{4 \pm \sqrt{16 - 4(3)(-5)}}{6}$

$x = \frac{4 \pm \sqrt{16 + 60}}{6}$

$x = \frac{4 \pm \sqrt{76}}{6}$

Now do Activity 1 Nos 8–9.

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

Solve by completing the square

1. $x^2 - 2x - 7 = 0$
2. $x^2 - 4\mathrm{x} + 7 = 0$
3. $p^2 - 3p + 1 = 0$
4. $2x^2 - 3x - 1 = 0$
5. $x^2 - 2x - 24 = 0$
6. $5x^2 - 3x - 1 = 0$
7. $5 - x - 2x^2 = 0$
8. $3x^2 - 6x - 2 = 0$
9. $9x^2 - 6mx + m^2 - 3m = 0$

Lesson 7

QUADRATIC FORMULA

Using the formula to solve quadratic equations

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and algebra
Assessment Standard
The solution of quadratic equations by using the formula.

Overview

In this lesson you will:

- Use the quadratic equation formula to solve quadratic equations.
- Read the questions carefully so as to decide whether to leave the answer in surd form or to use your calculator and write the answer in surd form.
- Learn to write the equation in the simplified surd form.
- Look at the formula and decide when you have solutions and when you do not.
- Look at the formula and decide whether you could have factorised or whether you had to use the formula to solve the equation.

DVD

Lesson

Using the formula to solve quadratic equations. Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

We look at $b^2 - 4ac$

Why is $b^2 - 4ac$ so important?

It is under a root, and we know that square roots of negative numbers do not exist in the real domain.

So if $b^2 - 4ac < 0$ there will be no real solution

$2x^2 + 4x + 3 = 0$

$b^2 - 4ac = 16 - 4(6)$

$b^2 - 4ac$ is Negative – so x is non real.

If $b^2 - 4ac$ is a perfect square we don't need to use a calculator. **Why? Because a perfect square can be square rooted** e.g. $\sqrt{25} = \sqrt{5^2} = 5$. So we can then factorise.

$2x^2 - 5x - 3 = 0$

$b^2 - 4ac = 25 - 4(2)(-3)$

$= 49$ Perfect square, so

$(2x + 1)(x - 3) = 0$

$x = \frac{-1}{2}$ or $x = 3$

So when do we need to apply the formula?

– When $b^2 - 4ac$ is a positive, non-perfect square

Like: $\sqrt{12}$ or $\sqrt{17}$ etc.

If $b^2 - 4ac$ is a non-perfect square, x is an irrational number.

Example 1

Solve for x and write your answer correct to two decimal places.

$2x^2 - 5x + 1 = 0$

Solution

$a = 2 \quad b = -5 \quad c = 1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \frac{5 \pm \sqrt{17}}{2(2)}$

$\therefore x = 5 \pm \frac{\sqrt{17}}{4}$

$x = 2{,}28 \quad \text{or} \quad x = 0{,}22$

Check $b^2 - 4ac$: $b^2 - 4ac = 25 - 4(2)(1)$

$= 25 - 8$

$= 17$

17 is not a perfect square, so we know that we need to use the formula.

Check $b^2 - 4ac$: $b^2 - 4ac = (-2)^2 - 4(1)(-1)$

$= 4 + 4$

$= 8$

8 is not a perfect square

Example 2

Solve for x and write your answer in the simplest surd form.

$x^2 - 2x - 1 = 0$

Solution

$\therefore x = \frac{-(-2) \pm \sqrt{8}}{2}$

$= \frac{2 \pm \sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

Example 3

Solve for x: $\quad -3x^2 + 2x - 5 = 0$: $b^2 - 4ac = 4 - (4)(-3)(-5)$

$= 4 - 60$

$= -56$

$b = \ 4ac$ is negative

Thus no real solution

Example 4

Solve for x: $\quad 6x^2 + 5x - 6 = 0$

$b^2 - 4ac = 25 - 4(6)(-6)$

$= 169$

169 is not a perfect square, so we can factor

Solutions

Method 1: We can factorise

$(2x + 3)(3x - 2) = 0$

$x = \frac{-3}{2} \ \text{or} \ x = \frac{2}{3}$

Method 2: Maybe we did not want to factorise, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ x = \frac{-5 \pm \sqrt{169}}{12}$$

$$\therefore\ x = \frac{-5 + 13}{12} \qquad \text{or} \qquad x = \frac{-5 - 13}{12}$$

$$\therefore\ x = \frac{8}{12} \qquad \text{or} \qquad x = \frac{-18}{12}$$

$$\therefore\ x = \frac{2}{3} \qquad \text{or} \qquad x = \frac{-3}{2}$$

So, when are there solutions and when are there no solutions, and when could you actually have factorised instead of using the formula?

Strange example

If the roots of $x^2 - mx + n = 0$ are consecutive integers, find m in terms of n.

$a = 1 \qquad b = -m \qquad c = n$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{m \pm \sqrt{m^2 - 4(n)}}{2}$$

If $R_1 = \frac{m + \sqrt{m^2 - 4n}}{2}$ and $R_2 = \frac{m - \sqrt{m^2 - 4n}}{2}$ then $R_1 \geq R_2$

So: $R_1 = R_2 + 1$ Consecutive numbers

$$\therefore\ \frac{m + \sqrt{m^2 - 4n}}{2} = \frac{m - \sqrt{m^2 - 4n}}{2} + 1$$

$$\therefore\ m + \sqrt{m^2 - 4n} - m + \sqrt{m^2 - 4n} = 2$$

$$2\sqrt{m^2 - 4n} = 2$$

$$\sqrt{m^2 - 4n} = 1$$

$$m^2 - 4n = 1$$

$$m^2 = 1 + 4n$$

$$m = \pm\sqrt{1 + 4n}$$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

Solve for *x*:

1. $9x(x - 1) = -2$
2. $2x^2 + 4x + 3 = 0$
3. $2x^2 + x - 2 = 0$
4. $\frac{1}{4}x^2 + \frac{1}{3}x - 7 = 0$
5. $24x^2b^2 + 2xb - 1 = 0$
6. $36x^4 - 25x^2 + 4 = 0$
7. $x^2(x + m) - 2x(x + m) + x + m = 0$
8. $\frac{1}{2}x^2 - 3x + 1{,}3 = 0$ (correct to 2 dec)
9. $(x + 3)(2 - x) = 7x$ (correct to 2 dec)

QUADRATIC AND EXPONENTIAL EQUATIONS

Using substitution methods

Lesson 8

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
Learning Outcome 2: Functions and Algebra
Assessment Standard

- Expressions and equations using the Laws of exponents for rational exponents.
- Quadratic equations.

Overview

In this lesson you will:

- Use a substitute variable to help us solve a quadratic or exponential equation.
- Mix concepts that we have covered in previous lessons.
- Practise mathematical modelling.

Lesson

LC DVD

Example 1

$(2x^2 + x)^2 - 5(2x^2 + x) + 6 = 0$

similar brackets
powers differ

Solution

$\therefore$ Let $k = 2x^2 + x$

Then $k^2 - 5k + 6 = 0$

$\therefore (k - 3)(k - 2) = 0$

$\therefore k - 3 = 0$ or $k - 2 = 0$

But $k = 2x^2 + x$: Then $2x^2 + x - 3 = 0$ or $2x^2 + x - 2 = 0$

$\therefore (2x + 3)(x - 1) = 0$

$\therefore 2x = -3$ or $x = 1$

$\therefore x = -\frac{3}{2}$

Example 2

$x^2 - 3x + \frac{40}{x(x-3)} = 14$: Solve for x

$\Rightarrow x^2 - 3x + \frac{40}{x(x-3)} = 14$

$\therefore x(x - 3) + \frac{40}{x(x-3)} - 14 = 0$ Restrictions: $x \neq 0$; $x \neq 3$

Let $k = x(x - 3)$

Then $k + \frac{40}{k} - 14 = 0$

Now: $k^2 - 14k + 40 = 0$ ($\times\, k$ and get into standard form)

$\therefore (k - 10)(k - 4) = 0$

Now: $x(x-3)-10=0$ and $x(x-3)-4=0$

$\therefore x^2-3x-10=0$ | $x^2-3x-4=0$

$\therefore (x+2)(x-5)=0$ | $\therefore (x+1)(x-4)=0$

$\therefore x=-2$ or $x=5$ | $\therefore x=-1$ or $x=4$

3. Solve for x: $\sqrt{x^2-x-3}-5=-x^2+x$

Solution

Create a k. Isolate the surd.

$\sqrt{x^2-x-3}-5=-(x^2-x)$

$\sqrt{(x^2-x)-3}=-(x^2-x)+5$

Let $x^2-x=k$

$\sqrt{k-3}=-k+5$

Restriction: $k-3\geq 0 \Rightarrow k\geq 3$

$-k+5\geq 0 \Rightarrow k\leq 5$

$\therefore 3\leq k\leq 5$

$k-3=(5-k)^2$

$k-3=25-10k+k^2$

$0=k^2-11k+28$

$0=(k-7)(k-4)$

$k=7$ or $k=4$ Check

$k\neq 7$ since $3\leq k\leq 5$

$x^2-x=4$

$x^2-x-4=0$

Check: $b^2-4ac=(-1)^2-4(1)(-4)$ So $k=4$ is only the solution

$=1+16$

$=17$

So $x=\dfrac{1\pm\sqrt{17}}{2}$

Now do Activity 1.

Solving exponential equations using substitution

Example 4

Solve for x: $5^{x+1}-24=5^{1-x}$

Solution

Split up the exponents. Make the exponents positive.

$5^x\cdot 5-24=5^1\cdot 5^{-x}$

$5\cdot 5^x-24=\frac{5}{5^x}$

let $5^x=k$; $k>0$

$5k-24=\frac{5}{k}$

$5k^2-24k=5$

$5k^2-24k-5=0$

$(5k+1)(k-5)=0$

$k\neq -\frac{1}{5}$ or $k=5$ $(k>0)$

invalid $5^x=5$

$x=1$

Why is $k>0$? Since we are substituting 5^x with k, and we know that $5^x>0$ for all x, it will have the consequence that $k>0$ since $k=5^x$

Example 5

Solve for x: $3^x + 3^{4-2x} = 1 + 3^{4-x}$

PAIRS

FORMATIVE ASSESSMENT

Solution

$3^x + 3^{4-2x} = 1 + 3^{4-x}$

$\therefore 3^x + \frac{3^4}{3^{2x}} = 1 + \frac{3^4}{3^x}$ (Remember $a^{-m} = \frac{1}{a^m}$)

$\times 3^{2x}$: $3^{3x} + 3^4 = 3^{2x} + 3^x.3^4$

Let $3^x = k \rightarrow$ then $3^{2x} = k^2$ and $3^{3x} = k^3$

$\therefore k^3 - k^2 - 3^4k + 3^4 = 0$

$\therefore k^2(k-1) - 3^4(k-1) = 0$

$\therefore k^2 = 3^4$ or $k = 1$

$\therefore 3^{2x} = 3^4$ $\quad 3^x = 1 = 3^0$

$2x = 4$ $\quad \therefore x = 0$

$x = 2$

Now do Activity 2.

Mathematical modelling

DVD

1. Factorise $1 - \frac{1}{m^2}$
2. Hence, without using a calculator, evaluate

   $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\ldots\left(1-\frac{1}{400}\right)$

Solution

1. $\left(1-\frac{1}{m}\right)\left(1+\frac{1}{m}\right)$ Now use in the 2nd part of the question
2. $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\ldots\left(1-\frac{1}{20}\right)\left(1+\frac{1}{20}\right)$

   Work out each bracket

   $\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right)\ldots\left(\frac{19}{20}\right)\left(\frac{21}{20}\right)$

   Notice that consecutive products cancel out one another

   $\left(\frac{1}{2}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{\cancel{4}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\ldots\left(\frac{\cancel{20}}{\cancel{19}}\right)\left(\frac{\cancel{19}}{\cancel{20}}\right)\left(\frac{21}{20}\right)$

   Answer:

   $\left(\frac{1}{2}\right)\left(\frac{21}{20}\right) = \frac{21}{40}$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

Solve for x:

1. $3x^2 - x + \frac{6}{3x^2 - x + 2} = 5$
2. $(x^2 - 5x)^2 - 2(x^2 - 5x) = 24$
3. $\sqrt{3x - 1} + 1 = \frac{6}{\sqrt{3x - 1}}$
4. $(x^2 - 3x)^2 - 5(x^2 - 3x) + 4 = 0$ (correct to 2 decimal places)
5. $4(3x^2 + x + 1) - \frac{10(3x^2 + x - 1)}{3x^2 + x} = 7$
6. $2x - 3 - \frac{3}{2x - 1} = 0$
7. $x^2 + 6x - 2 = \frac{35}{x^2 + 6x}$
8. $(2x^2 - 3x)^2 - 4x^2 + 6x - 3 = 0$
9. $\frac{\sqrt{2x - 1}}{2} - \frac{4}{\sqrt{2x - 1}} - 1 = 0$
10. $x^2 - \frac{5}{x^2 + x} = 4 - x$

Activity 2

Solve:

1. $3^{2x} - 12 \cdot 3^x + 27 = 0$
2. $3^{x+1} - 10 \cdot 3^{x-1} + 3 = 0$
3. $\left(2^{\frac{x}{2}} - 4 \cdot 2^{-\frac{x}{2}}\right)\left(2^{\frac{x}{2}} - 8 \cdot 2^{-\frac{x}{2}}\right) = 0$
4. $4 \cdot 2^{\frac{2x}{3}} - 33 + 8 \cdot 2^{-\frac{2x}{3}} = 0$
5. $4 \cdot 2^{2x+2} - 17 \cdot 2^x + 1 = 0$
6. $8 \cdot 2^{2x} + 10 \cdot 2^x - 3 = 0$

QUADRATIC EQUATIONS

Simultaneous equations and modelling

Lesson 9

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and algebra
Assessment Standard
Equations in two unknowns, one of which is linear and one of which is quadratic.
Use mathematical models to investigate problems that arise in real–life contexts.

Overview

In this lesson you will:

- Solve simultaneous equations.
- Form your own equation and use the techniques of simultaneous equations to solve problems.

Lesson

LC DVD

Example 1

Solve for x and y in

$x + 2y = 5$ and $2y^2 - xy - 4x^2 = 8$

$\Rightarrow$ We use: $x + 2y = 5$

(1) – $\therefore x = 5 - 2y$ (isolate x to avoid fraction)

Now substitute:

$2y^2 - (5 - 2y)y - 4(5 - 2y)^2 = 8$

$\therefore 2y^2 - 5y + 2y^2 - 4(25 - 20y + 4y^2) = 8$

$\therefore 4y^2 - 5y - 100 + 80y - 16y^2 - 8 = 0$

$\therefore -12y^2 + 75y - 108 = 0$

Thus

$12y^2 - 75y + 108 = 0$

$\therefore 4y^2 - 25y + 36 = 0$

$\therefore (4y - 9)(y - 4) = 0$

$\therefore y = \frac{9}{4}$ or $y = 4$

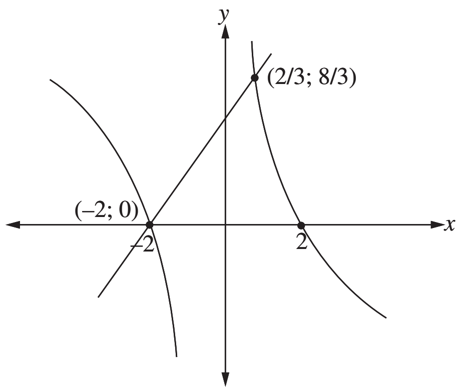
Now substitute back into (1)

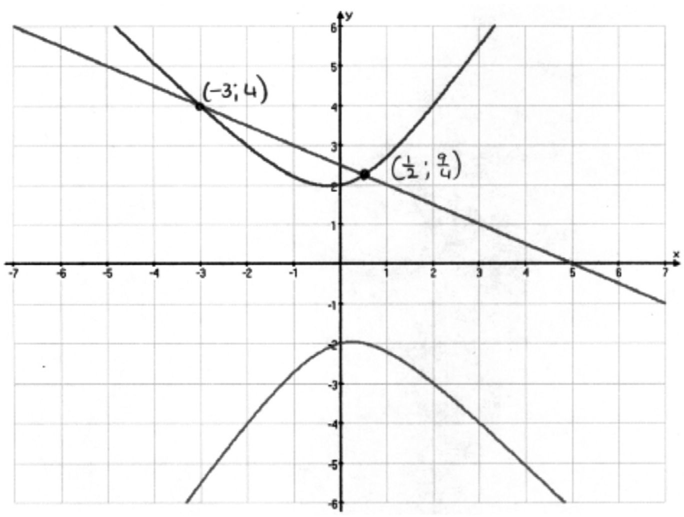
Then $x = 5 - 2\left(\frac{9}{4}\right)$ or	$x = 5 - 2(4)$
$x = 5 - \frac{9}{2}$	$x = 5 - 8$
$x = \frac{10 - 9}{2}$	$x = -3$
$x = \frac{1}{2}$	

$\therefore$ If $x = \frac{1}{2}$; $y = \frac{9}{4}$ and if $x = -3$; $y = 4$

Why solve simultaneous equations?

It is a method that helps us to determine where the graphs of $y - x = 2$ and $x^2 + 2xy - 4 = 0$ cut one another graphically





Example 2

Solve for x and y if

$x + y = 9$ and $x^2 - 4xy - y^2 = 8$ –(2)

(correct to two decimals)

$\Rightarrow x + y = 9$

Then $y = 9 - x$ –(1)

(2) $\mapsto$ (1) $x^2 - 4x(9 - x) - (9 - x^2) = 8$

$\therefore x^2 - 36x + 4x^2 - (81 - 18x + x^2) - 8 = 0$

$\therefore 5x^2 - 36x - x^2 + 18x - 81 - 8 = 0$

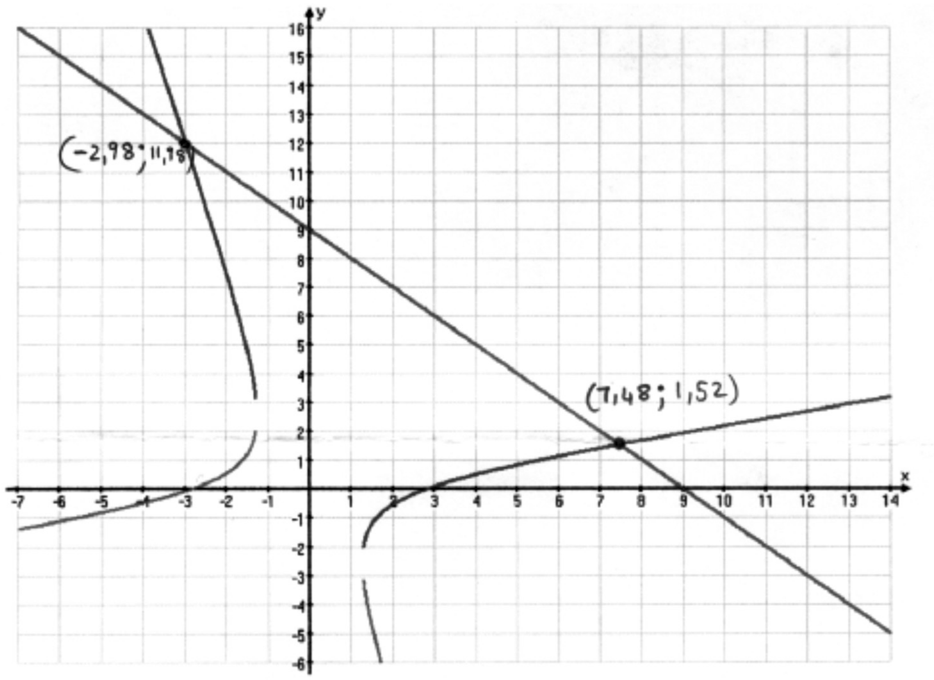
$\therefore 4x^2 - 18x - 89 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{18 \pm \sqrt{(18)^2 - 4(4)(-89)}}{2(4)}$

$x = \frac{18 \pm \sqrt{1\,748}}{8}$

$\therefore x = 7{,}48$ or $x = -2{,}98$



Then $y = 9 - (7{,}48)$ $y = 9 - (-2{,}98)$

$\therefore y = 1{,}52$ $y = 9 + 2{,}98$

$y = 11{,}98$

Thus: If $x = 7{,}48$; $y = 1{,}52$ and if $x = -2{,}98$; $y = 11{,}98$

Example 3

3. Solve for x and y: $3x - 4y = 3$ and $xy = 15$

$\Rightarrow$ We cannot avoid fractions:

$xy = 15$

Thus $x = \frac{15}{y}$ … (1) and $3x - 4y = 3$ … (2)

(1) $\rightarrow$ (2): $3\left(\frac{15}{y}\right) - 4y = 3$

$\therefore \frac{45}{y} - 4y = 3$

$\therefore 45 - 4y^2 = 3y$

$\therefore 4y^2 + 3y - 45 = 0$

$\therefore (4y + 15)(y - 3) = 0$

$\therefore y = -\frac{15}{4}$ or $y = 3$

Then

$x = \frac{15}{-\frac{15}{4}} = 15.-\frac{4}{15}$

$\therefore x = -4$ or $x = \frac{15}{3}$

$\therefore x = 5$

Example 4

Matthew Matic and Allen Algebra attended the polynomial pop concert in Trigoville.

“Do you know” said Matthew, “that if there were 60 people less attending this concert, then the audience could be arranged in a perfect square”.

“Yes” replied Allen, “and if there were 41 more people attending, then the length of the square would be increased by 1”.

How many people were attending this concert?

Solution

Let x be the number of people attending the concert.

Note: It is of utmost importance that you define your variable

$x - 60 = y^2$ (1)

$x + 41 = (y + 1)^2$ (2)

Make x the subject in equation 1

$x = y^2 + 60$

Substitute into equation 2

$y^2 + 60 + 41 = (y + 1)^2$

Solve

$y^2 + 101 = y^2 + 2y + 1$

$2y = 100$

$y = 50$

$x = y^2 + 60$

$x = 2\,500 + 60$

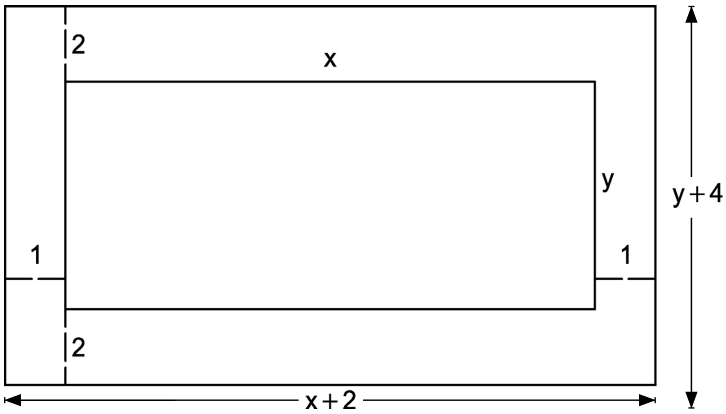
$x = 2\,560$

Always answer the question

Thus 2 560 people attended the concert.

Example 5

A card has 10 cm^2 of printing. The margins at the top and the bottom are 2 cm and along the sides are 1 cm. If the area of the whole card is 39 cm^2, find possible dimensions for the card.



Solution

$xy = 10$

$(x + 2)(x + 4) = 39$

$xy + 2y + 4x + 8 = 39$

Substitute $y = \frac{10}{x}$ into the above equation.

$x\left(\frac{10}{x}\right) + 2\left(\frac{10}{x}\right) + 4x + 8 = 39$

$10 + \frac{20}{x} + 4x + 8 = 39$

$10x + 20 + 4x^2 + 8x = 39$

$4x^2 - 21x + 20 = 0$

$(4x - 5)(x - 4) = 0$

$x = \frac{5}{4}$ or $x = 4$

$y = \frac{10}{\frac{5}{4}}$ $y = \frac{10}{4}$

$y = 8$ $y = \frac{5}{2}$

Dimensions

$x = 6$ $y = 6{,}5$ or $x = 1{,}23$ $y = 12$

Now do Activity 2.

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

In the following, solve for x and y

1. $x + y = 9$ and $x^2 + xy + y^2 = 61$
2. $x - 2y = 3$ and $4x^2 - 5xy = 3 - 6y$
3. $(x - 1)^2 + (y - 2)^2 = 5$ and $2x + y + 1 = 0$
4. $(x - 2)^2 + (y - 3)^2 = 0$
5. $(y - 3)(x^2 + 2) = 0$
6. $2x - 3y = 1$ and $x(x - 6) - y = (1 - y)(1 + y)$

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. A number of two digits is seven times the sum of its digits. The product of the digits is diminished by the tens digit, leaves twelve times the ten digits divided by the units digit. Find the number.
2. The product of two consecutive integers is 90. Find the integers.

Lesson 10

QUADRATIC EQUATIONS

Solving inequalities

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment Standard
Solving linear and quadratic equations.

Overview

In this lesson you will:

- Explore the number line to decide where inequalities are negative, zero or positive.
- Learn the correct notation for inequalities.
- Use a number line to solve quadratic inequalities.

Lesson

DVD

Look at this statement

$4 < 6$	True
Multiply by + 2	
$8 < 12$	True
Divide by + 4	
$2 < 3$	True
Multiply by -1	
$-2 < -3$	False
but $-2 > -3$	

Rule

When we multiply or divide an inequality by a negative number, the inequality changes direction.

Examples

1. **Solve: $3x + 6 < -3$**

Solution

$3x < -3 - 6$

$3x < -9$ Divide by +3

$x < -3$ Don't change the inequality sign

2. Solve: $4 - 2x < 5$

Solution

$-2x < 5 - 4$

$-2x < 1$

Divide both sides by -2 and change the inequality sign.

$x > -\frac{1}{2}$

Quadratic inequalities

- To solve a quadratic inequality, make the one side of the inequality equal to zero and x^2 has a positive coefficient.
- Factorise;
- Draw a number line;
- Decide whether the zeroes can be nought;
- Test the number line; and
- Write down the correct solution.

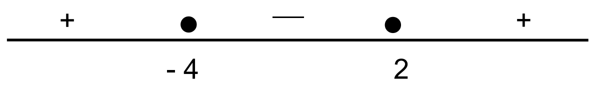
Rules

- Never multiply the denominator away.
- If you multiply or divide by a negative, change the inequality sign.
- Non-real numbers are square roots of negative numbers.
- An expression is undefined when the denominator is zero.
- Always make sure that every x has a positive coefficient, then the far right of the number line will be positive.

Look at the expression

$(x - 2)(x + 4)$

Explore the number line



Note

$(x - 2)(x + 4) = 0$

when $x = 2$ or $x = -4$

$(x - 2)(x + 4) > 0$

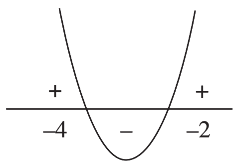
when $x < -4$ or $x > 2$

$(x - 2)(x + 4) < 0$

when $-4 < x < 2$

Why is this so?

If we look at the graph of $y = (x - 2)(x + 4)$, it looks like this:



So if $-4 < x < 2$ the y values on this graph are negative.

Always make sure that your x^2 term is positive, and that all x terms have positive coefficients. Then you will always have a + sign on the far RHS of the number line.

Rules

- Ensure that one side of the inequality is zero.
- Factorise
- Explore the number line
- Write down the correct solution.

Examples

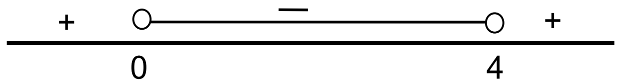
1. **Solve for x: $x^2 < 4x$**

Solution

Isolate the inequality

$x^2 - 4x < 0$ Make one side zero

$x(x - 4) < 0$ Factorise.



Write the zeroes on the number line.

We are looking for $x(x - 4) < 0$ (negatives on the number line).

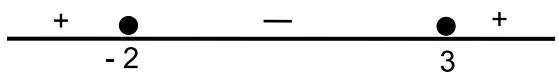
$0 < x < 4$

2. **Solve for x: $x^2 - x \geq 6$**

Solution

$x^2 - x - 6 \geq 0$

$(x - 3)(x + 2) \geq 0$



We are looking for positive on the number line.

$x \leq -2$ or $x \geq 3$

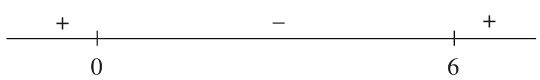
3. **Solve for x: $6x \leq x^2$**

Solution

$6x \leq x^2$ Make sure that x^2 has a positive sign

$\therefore x^2 - 6x \geq 0$ Do not use $6x - x^2 \leq 0$

$\therefore x(\text{x} - 6x \geq 0$



We are looking for negative

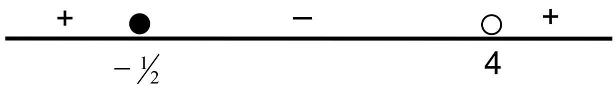
$x \leq 0$ or $x \geq 6$

4. Look at this one

Solve for x: $\dfrac{2x+1}{x-4} \geq 0$

Solution

Note : You already have factors and all the x's have positive coefficients.



Note: $-\frac{1}{2}$ can be zero but 4 cannot be zero.

$x \leq -\frac{1}{2}$ or $x > 4$

5. A strange one

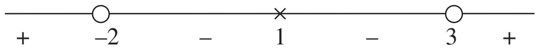
Solve for x: $\dfrac{(x-1)^2}{x^2-x-6} \geq 0$

Solution

We know that $x^2 \geq 0$ for all x
so $(x-1)^2 \geq 0$ for all x.

Furthermore: $x^2 - x - 6 = (x-3)(x+2)$

but $x \neq 3$ and $x = -2$ since they are both in the denominator. Because $(x-1)^2 \geq 0$, we are not going to change sign when we pass through 1.



So we get that

$x < -2$; $x = 1$; $x > 3$

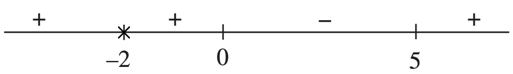
6. And this one

Solve for x: $\dfrac{5x-x^2}{(x+2)^2} \leq 0$

Solution

Let us change the sign:

$\dfrac{x^2-5x}{(x+2)^2} \geq 0$



$\therefore \dfrac{x(x-5)}{(x+2)^2} \geq 0$

$\therefore x < -2$; $-2 < x \leq 0$; $x \geq 5$

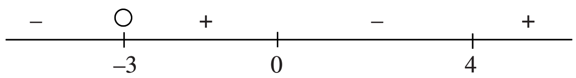
Application of inequalities

Examples

1. For which values of m is the following expression real: $\sqrt{\dfrac{m^2-4m}{m+3}}$

Solution

$\dfrac{m^2-4m}{m+3} \geq 0$



$\therefore \dfrac{m(m-4)}{m+3} \geq 0$

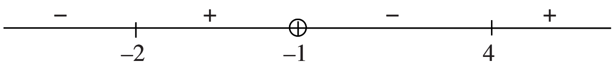
$-3 < m \leq 0$ or $m \geq 4$

2. For which values of k is the following expression undefined or non-real.
$\sqrt{\dfrac{(4-k)(2+k)}{(k+1)}}$

Solution

$\dfrac{(4-k)(2+k)}{k+1} < 0$

$\dfrac{(k-4)(k+2)}{k+1} > 0$



$-2 < k < -1$ or $k > 4$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

1. $x^2 - x > 0$
2. $x^2 - x < 6$
3. $x(x-2) \geq 8$
4. $4x - x^2 < 0$
5. $3x \geq x^2$

Activity 2

1. $x^3 - x^2 \geq 12x$
2. $\dfrac{4+x}{5-x} \leq 0$
3. $\dfrac{x^2-x-6}{x^2-x} \geq 0$
4. $\dfrac{2x-x^2}{x^2+7x+6} \leq 0$
5. $\dfrac{-x-4}{1-x^2} \geq 0$
6. $\dfrac{7x^2-14x}{1+x} > 0$

Example

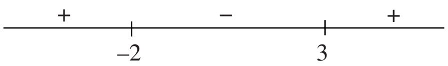
Solve $\dfrac{x^2+1}{x^2-x-6} < 0$

$x^2 \geq 0$ so $x^2 + 1 \geq 1$

So then for the outcome to be negative,

$x^2 - x - 6 < 0$

$\therefore (x-3)(x+2) < 0$



$-2 < x < 3$

Activity 3

1. For which values of k will the expression $\sqrt{9-k^2}$ be real.
2. For which values of p will the expression $\sqrt{\dfrac{3-p}{2+p}}$ be real
3. For which values of m will the expression $\sqrt{\dfrac{6+m-m^2}{m-1}}$ be non-real or invalid
4. For which values of t will $\sqrt{\dfrac{-(t-1)^2}{2t}}$ be real
5. For which values of p will $\dfrac{\sqrt{4-p^2}}{p-1}$ be real.

MATHEMATICAL MODELLING

Lesson 11

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and algebra
Assessment Standard
- Use mathematical models to investigate problems that arise in real-life contexts.
- Making conjecture, demonstrating and explaining their validity.
- Expressing and justifying mathematical generalisations of situations.
- Using the various representations to interpolate and extrapolate.

Overview

In this lesson you will:

- Solve the normal routine word sum problems.
- Look at non-routine questions involving proof.

For these problems we usually:

(1) State our variable.

(2) Organise the information in a table form if possible.

(3) Recognise our mathematical tools that apply in the organising equation.

(4) Solve our problem.

(5) Check our answer for validity.

Lesson

LC DVD

Routine problems

1. Area: The area of a room is 20 m^2.

If the length is increased by 3 m and the width by 1 m, the room will double in area. Determine the original dimensions of the room.

Example 1

Farmer Phillip has 1 500 trays of tomatoes which he can sell to the market today at R20 per tray. His neighbour informs him that he should wait a few days, as the market price will increase by about R2 per day. He knows that if he does this, he will lose about 5 trays of tomatoes per day due to them deteriorating. He asks you to help him by claculating if it will be a good idea to wait a few days longer.

Draw up a detailed explanation for him.

Solution

(1) Define the variable you will use:

Your choice of variables is very important. The longer he waits, the price increases, but the amount he can sell becomes less. So:

Let *x* be the number of days that he waits.

Then: Amount he has to sell after *x* days: $(1\,500 - 5x)$ trays

the amount of trays he started with

he loses 5 trays for each day he waits

His income per tray will be: $(20 + 2x)$

price he could get today

the price that increase with R2 per tray per day

His total income = Units sold × Price per unit

$= (1\ 500 - 5x)(20 + 2x)$

$I(x) = 30\ 000 + 2\ 900x - 10x^2$

So we can see that this is a quadratic equation, for which we can complete the square:

$-10\ (x^2\ - 290x - 3\ 000)$

$= -10\ (x^2\ - 290x + (145)^2 - (145)^2 - 3\ 000)$

$= -10[(x - 145)^2 - 24\ 025]$

$= -10(x - 145)^2 + 240\ 250$

So he has to wait 145 days, and then sell to the market to get an income of R240 250.

If he sells today, he will make R20 × (1 500) = R30 000

If he waits for 45 days:

He has (1 500 – 5(145)) trays = 775 trays

He can sell at (20 + 2(145)) = R310 per tray)

So his income will be R240 250

Example 2: Rate

Two machines, working together, complete a job in 2 hrs 24 min. Working on its own the one machine would take 2 hours longer than the other. How long does the slower machine take?

⇒ Let the slower machine take x hours

Then the faster machine will take $x - 2$ hours.

Together: 2,4 hours

$\therefore$ In 1 hour:

$\frac{1}{x} + \frac{1}{x-2} = \frac{1}{2{,}4}$

$\therefore (x - 2)2{,}4 + (x)2{,}4 = x(x - 2)$ … (multiply by LCD)

$\therefore 2{,}4x - 4{,}8 + 2{,}4x = x^2 - 2x$

$\therefore 4{,}8x - 4{,}8 = x^2 - 2x$

$\therefore 48x - 48 = 10x^2 - 20x$ … (multiply by 10 to rid decimals)

$\therefore 10x^2 - 68x + 48 = 0$

$\therefore 5x^2 - 34x + 24 = 0$

$\therefore (5x - 4)(x - 6) = 0$

$\therefore 5x = 4$ or $x = 6$

$x = \frac{4}{5}$ hours

impossible

Thus the slower machine takes 6 hours.

Example 3: Speed

A passenger train travels 10 km/h faster than a goods train. They have to travel from Station A to B, 100 km apart. It takes the goods 30 minutes longer than the passenger train to cover the distance. What is the speed of the goods train?

⇒ We want the speed of the goods train:

Let the speed be x:

	D	S	T
Goods	100	x	$\frac{100}{x}$
Passenger	100	$x + 10$	$\frac{100}{x + 10}$

D = Speed × Time

$\therefore t = \frac{D}{S}$

The determing factor here is time:-

$T_{GOODS} = T_{PASS} + 30$ min

$\therefore \frac{100}{x} = \frac{100}{x + 10} + 0{,}5$ (NB Speed in km/h, so time must be in hours)

$\therefore 100(x + 10) = 100x + \frac{1}{2}x(x + 10)$

$\therefore 200(x + 10) = 200x + x(x + 10)$

$\therefore 200x + 2\ 000 = 200x + x^2 + 10x$

$\therefore x^2 + 10x - 2\ 000 = 0$

$\therefore (x - 40)(x + 50) = 0$

$\therefore x = 40$ or $x = -50$

n.a.

Thus the speed of the goods train is 40 km/h.

Example 4: Financial

A person sets aside R1 800 for his holiday and budgets x rands per day. However, he spends R10 more per day and finds that he has to reduce his stay by 2 days.

3.1 Write down 2 expressions in x representing the number of days he actually spends on holiday.

3.2 Determine his budget expenditure per day whilst on holiday.

⇒ Total: R1 800 at Rx per day.

Now: $(x + 10)$ per day.

3.1 Planned days on holiday: $\frac{1\ 800}{x}$

Actual days on holiday: $\frac{1\ 800}{x + 10}$

$\therefore \frac{1\ 800}{x + 10} = \frac{1\ 800}{x} - 2$

Thus either $\frac{1\ 800}{x + 10}$ or $\frac{1\ 800}{x} - 2$

3.2 $\frac{1\ 800}{x + 10} = \frac{1\ 800}{x} - 2$

$\therefore 1\ 800x = 1\ 800(x + 10) - 2x(x + 10)$

$\therefore 1\ 800x = 1\ 800x + 18\ 000 - 2x^2 - 20x$

$\therefore 2x^2 + 20x - 18\ 000 = 0$

$\therefore x^2 + 10x - 9\,000 = 0$

$\therefore (x + 100)(x - 90) = 0$

$\therefore x = -100$ or $x = 90$

n.a.

$\therefore$ Budget expenditure per day – R90,00

Now do Activities 1 to 7

Non-routine problems and conjectures

- Select any three-digit number with all digits different from one another.
- Write down all possible two-digit numbers that can be formed from the three-digit number you chose.
- Divide the sum of the digit
- What is your answer.

Conjecturing and Proof

Do the following:

(1) Select a three digit number, with all digits different from one another and no digit equal to zero.

(2) Write down all the possible two digit numbers that can be formed from the three digits in your chosen number.

(3) Now divide the sum of the two digit numbers you formed, by the sum of the digits in your orginal three digit number. Write the answer down.

(4) Now repeat the first three steps for two more three digit numbers, and write the answers down.

(5) What do you notice?

(6) Formulate a conjecture from your observation.

(7) Now use algebraic methods to prove your conjecture true.

(8) If there is a zero digit, what will the quotient always be?

Steps (1 – 4)

$\Rightarrow$ The number: 123

Six possibilities: 12; 13; 23; 32; 31; 21

Sums: $\frac{12 + 13 + 23 + 32 + 31 + 21}{1 + 2 + 3} = \frac{132}{6} = 22$

The number: 479

Six possibilities: 47; 49; 79; 97; 94; 74

Sums: $\frac{47 + 49 + 79 + 97 + 94 + 74}{4 + 7 + 9} = \frac{440}{20} = 22$

The number: 659

Six possibilities: 65; 59; 69; 95; 56; 96

Sums: $\frac{440}{6 + 5 + 9} = \frac{440}{20} = 22$

(5) You will notice that your solution is always 22.

(6) If we take a three digit number and calculate the sum of all the two digit numbers that can be formed by using the digits in the three digit number,

and we divide this sum by the sum of the digits in the 3 digit number, the result will always be 22.

(7) Proof

Let the three digit number be *abc*. According to our base 10 place value system, this means that the number is:

$abc \rightarrow a\,(100) + b(10) + c = 100a + 10b + c$

The two digit numbers will be:

$10a + b$; $10a + c$; $10b + c$; $10c + b$; $10c + a$; $10b + a$

The sums:

$$\frac{10a + b + 10a + c + 10b + c + 10c + b + 10c + a + 10b + a}{a + b + c}$$

$$= \frac{20a + 2a + 20b + 2b + 20c + 2c}{a + b + c}$$

$$= \frac{22a + 22b + 22c}{a + b + c}$$

$$= \frac{22(a + b + c)}{a + b + c}$$

$= 22$

(8) $a\,0\,b \Rightarrow 100a + 0 + b$

$10a + 0$; $10a + b$; $10b + 0$; $10b + a$

Sums: $\dfrac{10a + 0 + 10a + b + 10b + 0 + 10b + a}{a + b}$

$$= \frac{20a + a + 20b + b}{a + b}$$

$$= \frac{21a + 21b}{a + b}$$

$= 21$

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. A printed rectangular card is to have a perimeter of 42 cm. The printed area is to be 50 cm^2. The margins above and below in the length of the card are to be 1 cm. The side margins are each to be 2 cm. Find the dimensions of the card.

2. The difference between two natural numbers is 3 and their product is 40. Calculate the numbers.

3. If the speed of a new locomotive was 10 km/h faster than an older model, then the journey between two stations 100 km apart would be shortened by 30 minutes. Determine the speed of the old model.

4. An electrician and his apprentice, working together, complete the wiring of a building in 20 days. If the apprentice were to do the work on his own, he would take 9 days longer than the electrician. How long would it take each one of them if they did the work on their own.

5. A boat can travel 30 km/h in still water. The boat takes one hour to travel 12 km upstream and back. Find the speed at which the stream is flowing.

6. A train which is x km long normally takes $\frac{5x}{2}$ minutes to pass through a tunnel which is 1 500 m long. Due to heavy rains it is forced to reduce its normal speed by 20 km/h and takes $\frac{15x}{4}$ minutes to pass through the tunnel. Find the length of the train.

7. The denominator of a positive fraction is one more than the square of the numerator. If the numerator is increased by 1 and the denominator decreased by 3, the value of the new fraction is $\frac{1}{4}$. Find the original fraction.

8. a) A group of people, consisting of x persons, must share an amount of R120 equally. How much must each person pay. (in terms of x)?

   b) Four of these persons refuse to pay. How much must the rest each pay to square the amount (in terms of x)?

   c) If the remainder of the group (i.e. those who are prepared to pay) must each pay R5 more than their original share, calculate how many people there were in the original group.

9. A man bought a number of sheep for R960. He lost four, but by selling the rest at a profit of R8 a piece, he made good his loss. How many sheep did he buy?

TRIGONOMETRIC FUNCTIONS

Lessons 12 – 13

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
The learner is able to investigate, analyse, describe and represent a wide range of functions and solve relatedproblems.
Assessment Standard
Derive and use the values of trigonometric functions.

Overview

In this lesson you will:

- Look at angles bigger than 90° and smaller than zero.
- Define the sine, cosine and tangent ratios in terms of the coordinates of a point and the radius of a circle.
- Establish the sign of the three trigonometric ratios in different quadrants.
- Use Pythagoras' theorem to find the values of the three different sides.
- Consider angle as a revolution instead of just in a shape.

Lesson

LC DVD

Prior knowledge

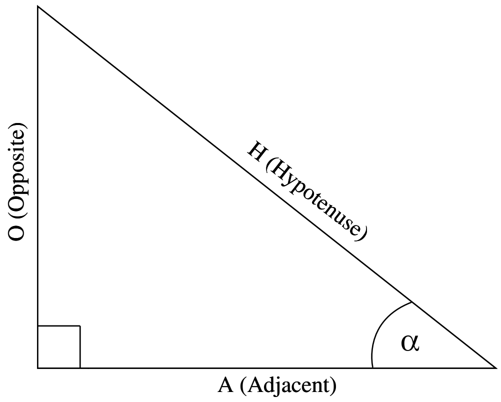
In Grade 10 we learnt:

$\sin \alpha = \frac{O}{H}$

angle ratio

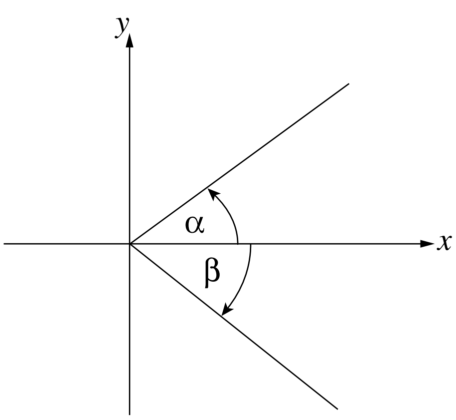
$\cos \alpha = \frac{A}{H}$

$\tan \alpha = \frac{O}{A}$



Part One

Bigger and smaller angles



If an angle is measured anti-clockwise it is positive.

If an angle is measured clockwise it is negative.

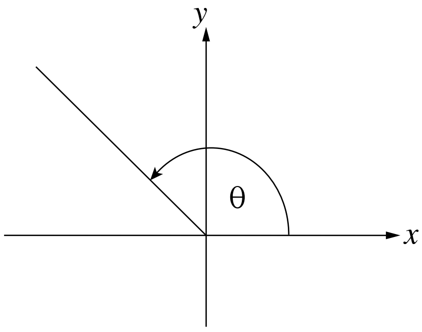
so $\alpha = 30°$

or $\alpha = 390°$

or $\alpha = -330°$

or $\alpha = -690°$

If $\beta = -40°$ or $\beta = 320°$ or $\beta = 680°$
we call these co-terminal angles.

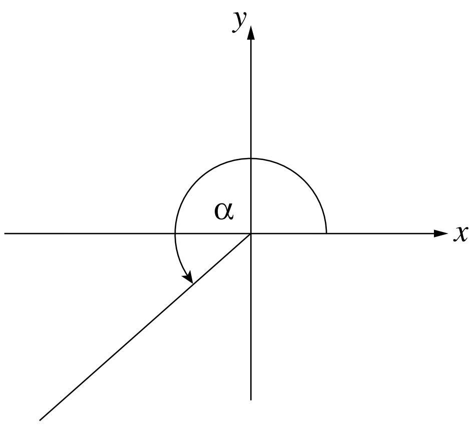


θ is in the second quadrant

If $\theta = 120°$ it could be

$\theta = -240°$

or $\theta = 480°$



If $\alpha = 200°$ it could be

$\alpha = -160°$

or $\alpha = 560°$

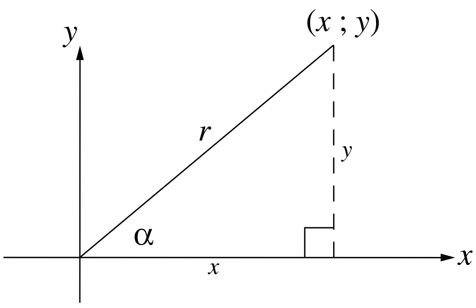
In which quadrant do you find:

1.	300°	4th
2.	(–150°)	3rd
3.	472°	2nd
4.	884°	2nd
5.	1 740	4th
6.	(–1 010)	1st

Now we will put the triangle into the Cartesian plane.

The first quadrant

Here $x > 0$ and $y > 0$. Remember that r will always be positive.



The angles are in the first quadrant

$\sin \alpha = \frac{y}{r} > 0$

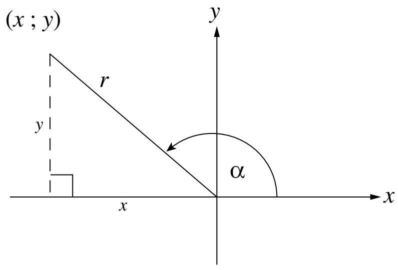
$\cos \alpha = \frac{x}{r} > 0$

$\tan \alpha = \frac{y}{x} > 0$

In the 1st quadrant, all ratios are positive.

The second quadrant

Here $x < 0$ and $y > 0$; $r > 0$



$\sin \alpha = \frac{y}{r} > 0$

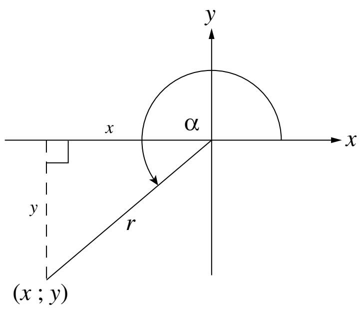
$\cos \alpha = \frac{x}{r} < 0$

$\tan \alpha = \frac{y}{x} > 0$

In the 2nd quadrant, sin α is positive but cos α and tan α are negative.

The third quadrant

Here $x < 0$ and $y < 0$; $r > 0$



$\sin \alpha = \frac{y}{r} < 0$

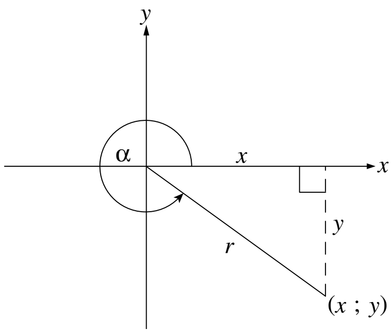
$\cos \alpha = \frac{x}{r} < 0$

$\tan \alpha = \frac{y}{x} > 0$

In the 3rd quadrant tan α is positive but sin α and cos α are negative.

The fourth quadrant

Here $x > 0$ and $y < 0$; $r > 0$



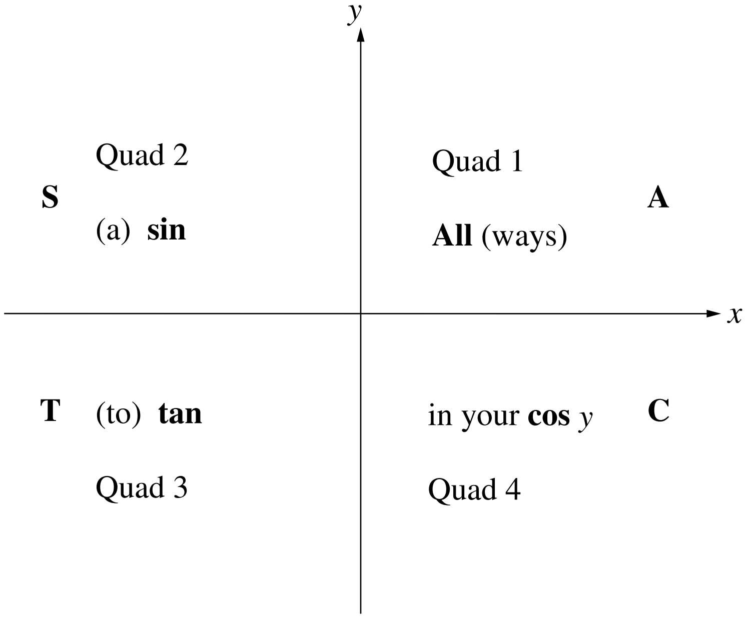
$\sin \alpha = \frac{y}{r} < 0$

$\cos \alpha = \frac{x}{r} > 0$

$\tan \alpha = \frac{y}{x} > 0$

In the 4th quadrant cos α is positive but sin α and tan α are negative.

Summary



In the 1st quadrant, **all** ratios are positive.

In the 2nd quadrant, **sin** α is positive but cos α and tan α are negative.

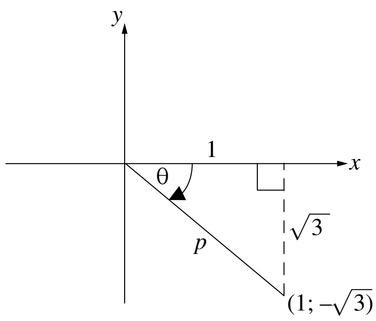
In the 3rd quadrant **tan** α is positive but sin α and cos α are negative.

In the 4th quadrant **cos** α is positive but sin α and tan α are negative.

Part Two:

Example 1

Calculate the value of sin θ, cos θ and tan θ if the terminal ray of θ is in standard position and passes through $(1; -\sqrt{3})$



$p^2 = (-\sqrt{3})^2 + (1)^2 = 3 + 1 = 4$ (Pythagoras)

$\therefore\ p = 2\ (p > 0)$.

So $\sin\theta = \frac{-\sqrt{3}}{2}$; $\tan\theta = -\sqrt{3}$

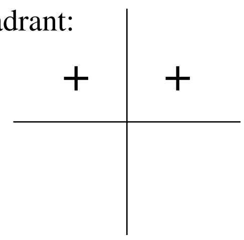
$\cos\theta = \frac{1}{2}$

Example 2

If $13 \sin\theta = 12$ and $\theta\ \varepsilon\ [90°\ ;\ 270°]$ find $\cos\theta \tan\theta$.

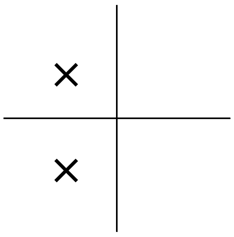
Locate the quadrant:

$\sin\theta = \frac{12}{13} > 0$

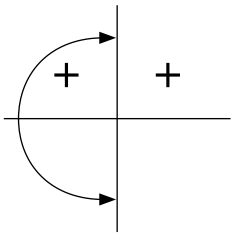


Since $\theta \in [90°; 270°]$

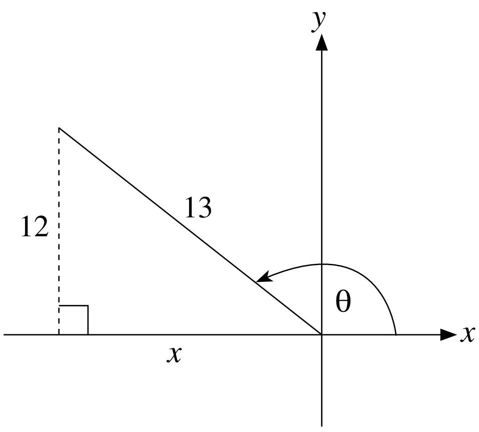
The overlap is in quadrant 2.



Place both on one sketch:



overlap in
quadrant 2



Pyth:

$x^2 = 13^2 - 12^2$

$= 169 - 144$

$= 25$

$\therefore$ x = –5 ($x < 0$)

Now:

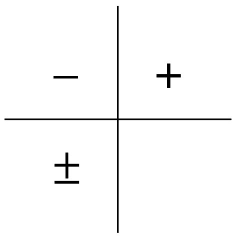
$\cos\theta \times \tan\theta = \left(\frac{-5}{13}\right)\left(\frac{12}{-5}\right) = \frac{12}{13}$

Example 3

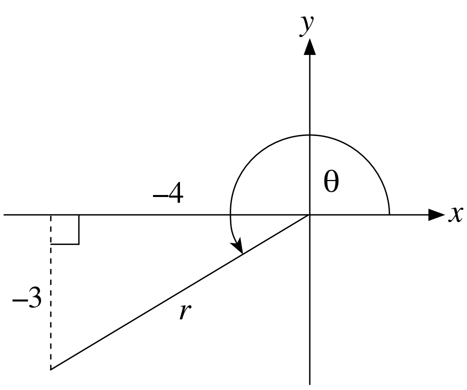
If 4 tan θ = 3 and cos θ < 0 find 2 sin θ - cos θ

$\tan\theta = \frac{3}{4} > 0$

$\cos\theta < 0$



Overlap in 3rd quadrant



$r = 5$ (Pythagorean triplet)

So 2sin θ – cos θ

$= 2\left(-\frac{3}{5}\right) - \left(-\frac{4}{5}\right)$

$= -\frac{6}{5} + \frac{4}{5}$

$= -\frac{2}{5}$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

1. Locate the quadrant where…
   a) $\tan\theta < 0$ and $\sin\theta > 0$
   b) $\cos\theta < 0$ and $\sin\theta > 0$
   c) $\theta \in [180°; 360°]$ and $\cos\theta < 0$
   d) $\sin\theta < 0$ and $\theta \in [0°; 270°]$
   e) $\tan\theta > 0$ and $\cos\theta > 0$
   f) $\tan\theta > 0$ and $\theta \in [90°; 270°]$?
2. If P(–3 ; 1) is a point in the Cartesian plane and $X\hat{O}P = \theta$. Find
   a) $\sin\theta + \cos\theta$
   b) $1 + \tan^2\theta$
   c) $\sin^2\theta + \cos^2\theta$

   (If necessary leave your answer in surd form)
3. If $12\tan\theta = -5$ find the value of (a) $\sin\theta$ (b) $\cos\theta$
4. If $25\cos x = -24$ and x ε [0° ; 180°], find the value of $\frac{2\tan x}{\sin x}$
5. If $41\cos\theta = -40$ and $\sin\theta > 0$, find the value of
   a) $1 + \tan^2\theta$
   b) $2\sin\theta - \cos\theta$
6. If $\sin\theta = p$ and $p < 0$ and θ ε [90° ‘; 270°], find the value of $\cos\theta$.
7. If $\tan\theta = mm > 0$ and θ ε [0° ; 90°], find the value of $\sin^2\theta - \cos^2\theta$
8. In △ABC $\hat{B}$ is an obtuse angle. $\sin A = \frac{3}{5}$ $\sin B = \frac{4}{5}$, find $\dfrac{\cos A + \cos B}{\tan A - \frac{1}{\tan B}}$
9. If $\sin x = \frac{2m}{1+m^2}$ and x ε [0° ; 90°] , find $\frac{1}{\cos x} + \tan x$.
10. If $\tan\theta = \frac{2xy}{x^2 - y^2}$ and θ ε [0° ; 90°] find $\sin\theta + \cos\theta$
11. If $\cos\theta = p$ and $p < 0$ and θ ε [180° ; 360°] find $\tan\theta$

TRIGONOMETRIC IDENTITIES

Lesson 14

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard
Derive and use the following identities:
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = 1$

Overview

In this lesson you will:

- Use *x*, *y* and *r* to derive the above two identities.
- Use the above identities to simplify trigonometric expressions.
- Use the above identities to prove more complicated trigonometric identities.

Lesson

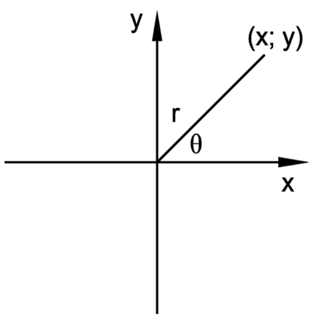
To simplify or prove trig expressions or identities, we need to change everything to sin θ and/or cos θ.

1. $\tan\theta = \frac{y}{x}$

$$\text{and } \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r}\times\frac{r}{x} = \frac{y}{x}$$

$$\text{so } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

2. The magic **ONE** of trigonometry



By Pythagoras $\quad x^2 + y^2 = r^2$

$\div r^2 \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$

but $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$

so $\quad \cos^2\theta + \sin^2\theta = 1$

Very important: $\sin^2\theta = 1 - \cos^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$

Let's use what we have learnt

1. Simplify: $\frac{1}{\cos^2\theta}(1 - \sin^2\theta)$

Solution

$\frac{1}{\cos^2\theta}\times\cos^2\theta = 1$ (Using $1 - \sin^2\theta = \cos^2\theta$)

2. Prove: $\frac{\sin\theta}{1+\cos\theta} + \frac{1}{\tan\theta} = \frac{1}{\tan\theta \times \cos\theta}$

Solution

When we prove an identity we have to show that what is on the LHS is exactly what is written on the RHS. So sometimes we need to simplify both to show this. Remember to keep the sides separate.

Prove that $\frac{1}{\cos x} + \tan x = \frac{\cos x}{1 - \sin x}$

LHS: $\frac{1}{\cos x} + \frac{\sin x}{\cos x}$

$= \frac{1+\sin x}{\cos x} \times \frac{1-\sin x}{1-\sin x}$

$= \frac{1-\sin^2 x}{\cos x(1-\sin x)}$

$= \frac{\cos^2 x}{\cos x(1-\sin x)} = \frac{\cos x}{1-\sin x}$

$\therefore$ LHS $\equiv$ RHS

3. Prove that $\frac{\sin\theta \cdot \tan\theta}{1-\cos\theta} - 1 = \frac{1}{\cos\theta}$

Proof:

LHS $= \frac{\sin\theta \cdot \frac{\sin\theta}{\cos\theta}}{1-\cos\theta} - 1$

$= \frac{\sin^2\theta}{\cos\theta(1-\cos\theta)} - 1$

$= \frac{(1-\cos^2\theta)}{\cos\theta(1-\cos\theta)} - 1$

$= \frac{(1-\cos\theta)(1+\cos\theta)}{\cos\theta(1-\cos\theta} - 1$

$= \frac{1+\cos\theta}{\cos\theta} - \frac{\cos\theta}{\cos\theta}$

$= \frac{1+\cos\theta-\cos\theta}{\cos\theta}$

$= \frac{1}{\cos\theta}$

= RHS

4. Prove $\left(\frac{1}{\cos x} - \tan x\right)^2 = \frac{1-\sin x}{1+\sin x}$

Solution

LHS $\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2$	$(\tan x = \frac{\sin x}{\cos x})$
$= \left(\frac{1-\sin x}{\cos x}\right)^2$	(LCD $\cos x$)
$= \frac{(1-\sin x)^2}{1-\sin^2 x}$	$(\cos^2 x = 1 - \sin^2 x)$
$= \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$	$(a^2 - b^2 = (a-b)(a+b))$
$= \frac{1-\sin x}{1+\sin x}$	

$\equiv$ RHS

5. Prove:

a) $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = \dfrac{2}{\sin A}$

b) Hence evaluate $\dfrac{\sin 2007°}{1 + \cos 2007°} + \dfrac{1 + \cos 2007°}{\sin 2007°}$

Solutions

a) $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = \dfrac{2}{\sin A}$

$$\text{LHS} \quad \frac{\sin^2 A + (1 + \cos A)^2}{\sin A(1 + \cos A)}$$

$$= \frac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{\sin A(1 + \cos A)}$$

$$= \frac{2 + 2\cos A}{(\sin A)(1 + \cos A)}$$

$$= \frac{2(1 + \cos A)}{\sin A(1 + \cos A)}$$

$$= \frac{2}{\sin A} = \text{RHS}$$

b) We proved $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = \dfrac{2}{\sin x}$

so $\dfrac{\sin 2007°}{1 + \cos 2007°} + \dfrac{1 + \cos 2007°}{\sin 2007°}$

$= \dfrac{2}{\sin 2007°}$

$= -4{,}405$

Activity

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. Simplify $\dfrac{\tan^2\theta - 1}{\cos^2\theta}$
2. Simplify $1\ ^{\circ}\ \sin^2\theta + \cos^2\theta$
3. Simplify $\dfrac{1}{\sin A} - \dfrac{\cos A}{\tan A}$
4. Simplify $\tan^2\alpha\cos^2\alpha + \dfrac{\sin^2\alpha}{\tan^2\alpha}$
5. Prove $\dfrac{\sin A - \sin^3 A}{\cos A - \cos^3 A} = \dfrac{1}{\tan A}$
6. Prove $\sqrt{1 + 2\sin\theta\cos\theta} = \sin\theta + \cos\theta$
7. Prove $\dfrac{1 - \tan x}{1 + \tan x} = \dfrac{\cos x - \sin x}{\cos x + \sin x}$
8. Prove $\dfrac{\tan^2\alpha}{1 + \tan^2\alpha} = \sin^2\alpha$
9. Prove $\left(\dfrac{1}{\sin x} - \sin x\right)^2 = \dfrac{1}{\tan^2 x} - \cos^2 x$
10. Prove $\dfrac{1 - 2\cos^2\theta}{\cos\theta\sin\theta} = \tan\theta - \dfrac{1}{\tan\theta}$
11. Prove $\dfrac{1}{\sin^2 A} - \dfrac{1}{\tan A\sin A} = \dfrac{1}{1 + \cos A}$
12. Prove $\tan^2\alpha\ ^{\circ}\ \sin^2\alpha = \tan^2\alpha \times \sin^2\alpha$
13. Prove $\dfrac{\sin^2\alpha}{1 + \cos\alpha} = 1 - \cos\alpha$
14. Prove $\dfrac{\sin x}{\cos x} = \dfrac{1 + \cos x}{\sin x}$

Lesson 15

REDUCTION FORMULA

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard
Derive the reduction formulae for:
$\sin(90^\circ \pm \alpha)$, $\cos(90^\circ \pm \alpha)$
$\sin(180^\circ \pm \alpha)$ $\cos(180^\circ \pm \alpha)$ $\tan(180^\circ \pm \alpha)$
$\sin(360^\circ \pm \alpha)$ $\cos(360^\circ \pm \alpha)$ $\tan(360^\circ \pm \alpha)$
$\sin(-\alpha)$ $\cos(-\alpha)$ $\tan(-\alpha)$

Overview

In this lesson you will:

- Learn to reduce all angles to co-terminal angles in the first quadrant.
- Simplify trigonometric expressions by writing ratios in terms of $\sin \alpha$ and $\cos \alpha$.
- Prove more trigonometric identities by examining the left-hand side and the right-hand side.

DVD

Lesson

The horizontal reduction formulae:

	90°	
	Sin	All
	$(180^\circ - \alpha)$	$(\alpha < 90^\circ)$
180°			$\frac{0^\circ}{360^\circ}$
	$(180^\circ + \alpha)$	$(360^\circ - \alpha)$
	Tan	Cos
	270°	

Here we look at angles in terms of the horizontal line 180°/360°.

Remember that the CAST rule still applies in the quadrants.

So every angle will be reduced by this horizontal reduction formulae to an angle that lies in the first quadrant. We do this by looking at the CAST rule, and the size of the angle.

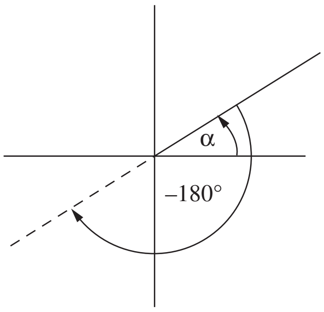
Let's try some:

- $\sin 125^\circ$ – (125° lies in the second quadrant)
  $= \sin(180^\circ - 55^\circ)$ – (the horizontal reduction formula in the 2nd Q)
  $= \sin 55^\circ$ (since the CAST rule says that sin is positive in 2nd Q)
- $\cos(180^\circ + \theta)$ – ($180^\circ + \theta$ lies in the 3rd Q and cos is negative here)
  $= -\cos\theta(180^\circ - \theta)$ – ($180^\circ - \theta$ in 2nd Q; tan negative here)
  $= -\tan\theta$
- $\sin(180^\circ - \theta)$ – ($180^\circ - \theta$ in 2nd Q; sin is positive here)
  $= \sin\theta$

Thinking of negative angles:

How do we measure the angle $(\alpha - 180^\circ)$? Instead of learning them by rote, let us unpack them visually.

We know positive angles are measured anti-clockwise, and negative angles are measured clockwise. So $\alpha - 180^\circ$ will be: α = anti-clockwise, then -180° becomes 180° clockwise.



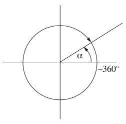
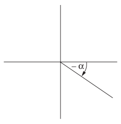
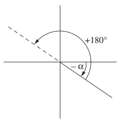
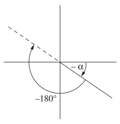
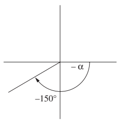
This tells us we are in the 3rd quadrant. Here tan is positive!

So $\sin(\alpha - 180°) = -\sin\alpha$

and $\cos(\alpha - 180°) = -\cos\alpha$

$\tan(\alpha - 180°) = \tan\alpha$

Let's try more:

- $\tan 214° \rightarrow$ (214° is in 3rd Q: 180° + 34 by horizontal reduction)
  $= \tan(180° + 34°)$ (In 3rd Q tan is positive)
  $= \tan 34°$
- $\sin(\alpha - 360°) \rightarrow$  – which is in the first Q
  $= \sin\alpha$
- $\tan^2(-\alpha) \rightarrow$  – in the 4th Q: tan is negative
  $= (-\tan\alpha)^2 \rightarrow$ anything that is squared is positive
  $= \tan^2\alpha$
- $\cos(-\alpha + 180°) \rightarrow$  2nd Q; cos is negative
  $= -\cos\alpha$
- $\sin(-\alpha - 180°) \rightarrow$  2nd Q; sin is positive
  $= \sin\alpha$
- $\tan(-150°) \rightarrow$  3rd Q; reduction (180° + α);
  ; tan is negative
  ; 30° away from – 180°

  $= -\tan 30°$

Example 1

Simplify the following

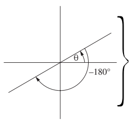
$$\frac{\cos(180° + \theta)\sin(\theta - 180°)}{\cos(-\theta)\tan(180° + \theta)}$$

Solution

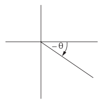
We look at one factor at a time to make sense of each one

(1) $\cos(180° + \theta) \rightarrow$ (180° + θ in 3rd Q → here cos is negative)

$= -\cos\theta$

(2) $\sin(\theta - 180°) \rightarrow$  3rd Q where sin is negative

$= -\sin\theta$

(3) $\cos(-\theta) \rightarrow$  4th Q where cos is positive

$= \cos\theta$

(4) $\tan(180° + \theta) \rightarrow$ 3rd Q where tan is positive

So: $\dfrac{\cos(180 + \theta)\sin(\theta - 180°)}{\cos(-\theta)\cdot\tan(180° + \theta)} = \dfrac{(-\cos\theta)(-\sin\theta)}{(\cos\theta)(\tan\theta)}$

$= \dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$ $\quad\left(\tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$

$= \cos\theta$

Now do Activity 1 no A.

REMEMBER

Ask yourself:

- *Which quadrant am I in?*
- *What is the sign of the ratios in the quadrant?*

DVD

Example 2

Prove $\dfrac{\cos x}{\sin x} = \dfrac{2\sin^2(180° + x)}{2\tan(180° + x) + 2\sin(-x)\cos x}$

Solution

RHS $\dfrac{2\sin^2 x}{2\tan x + 2(-\sin x)(\cos x)}$

$= \dfrac{2\sin^2 x}{1} \div \left(\dfrac{2\sin x}{\cos x} - \dfrac{2\sin x\cos x}{1}\right)$

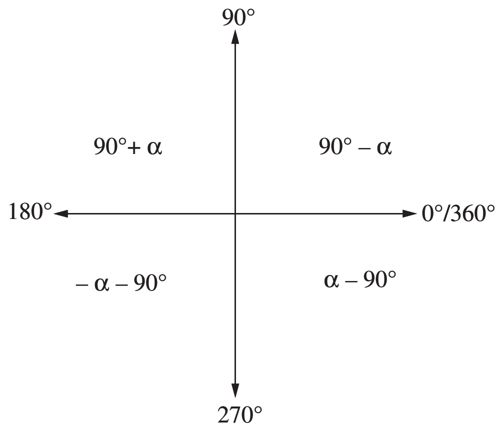
$= \dfrac{2\sin^2 x}{1} \div \dfrac{2\sin x - 2\sin x\cos^2 x}{\cos x}$

$= \dfrac{2\sin^2 x}{1} \times \dfrac{\cos x}{2\sin x(1 - \cos^2 x)}$

$= \dfrac{\cancel{2}\cancel{\sin^2} x}{1} \div \dfrac{\cos x}{\cancel{2}\sin x\cdot\cancel{\sin^2} x} = \dfrac{\cos x}{\sin x} = \text{LHS}$

Now do Activity 1 no B.

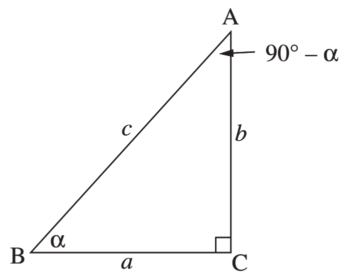
Vertical Reduction formulae



As the name suggests, we reduce angles in terms of the vertical line.

The CAST rule still applies here, but we now have to work with the complementary ratios.

Here is how they work:



In $\triangle ABC$, $\hat{C} = 90°$ has been given. So $\hat{A} + \hat{B} = 90°$ since all angles in a triangle add to 180°. We call $\hat{A}$ and $\hat{B}$ complements of one another, or we say they are complimentary angles.

Also notice that $\sin\alpha = \dfrac{b}{c} = \cos(90° - \alpha)$
$\Rightarrow \sin\alpha = \cos(90° - \alpha)$

and $\cos\alpha = \dfrac{a}{c} = \sin(90° - \alpha) \Rightarrow \cos\alpha = \sin(90° - \alpha)$

Outside of the triangle we will see that for angles expressed as a vertical reduction:

sine becomes cosine and cosine becomes sine

So $\sin \leftrightarrow \cos$

Let's try some:

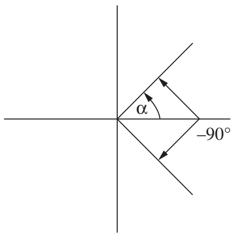
- $\sin(90^{\circ} + \alpha)$ $\rightarrow$ (($90^{\circ} + \alpha$) in 2nd Q: sin is positive here; because of the 90° sin becomes cos.)

  $= \cos \alpha$

- $\cos(90^{\circ} + \alpha)$ $\rightarrow$ (($90^{\circ} + \alpha$) in 2nd Q: cos is negative here; cos becomes sin because of 90°.)

  $= \sin \alpha$

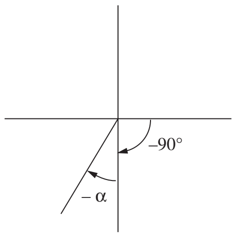
- $\cos(\alpha - 90^{\circ})$ $\rightarrow$



4th Quadrant; cos is positive here and cos becomes sin

  $= \sin \alpha$

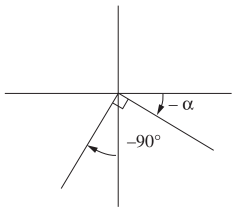
- $\cos(-90^{\circ} - \alpha)$ $\rightarrow$



in the 3rd Q: cos is negative and cos becomes sin

  $= -\sin \alpha$

- $\sin(-\alpha - 90^{\circ})$ $\rightarrow$



3rd Q: sun is negative and sin $\rightarrow$ cos

  $= \cos \alpha$

Example 3

Simplify $\dfrac{\tan(180^{\circ} + x)\cos(90^{\circ} - x)}{\sin(90^{\circ} - x)} - \dfrac{\cos(180^{\circ} - x)}{\sin(90^{\circ} + x)}$

Again: $\tan(180^{\circ} + x) \rightarrow$ [3rd Q: tan positive] $= \tan x$

$\cos(90^{\circ} - x) \rightarrow$ [1st Q: cos positive; cos $\rightarrow$ sin] $= \sin x$

$\sin(90^{\circ} - x) \rightarrow$ [1st Q: sin positive; sin $\rightarrow$ cos] $= \cos x$

$\cos(180^{\circ} - x) \rightarrow$ [2nd Q: cos negative] $= -\cos x$

$\sin(90^{\circ} + x) \rightarrow$ [2nd Q: sin positive; sin $\rightarrow$ cos] $= \cos x$

$= \dfrac{(\tan x)(\sin x)}{\cos x} - \dfrac{(-\cos x)}{\cos x}$

$= \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin x}{\cos x} + 1$ $\qquad \left(\tan x = \dfrac{\sin x}{\cos x}\right)$

$= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$ $\qquad (\sin^2 x + \cos^2 x = 1)$

Example 4

Prove that $\dfrac{\cos^2(90^\circ - x) + \sin^2(90^\circ + x)}{\tan(180^\circ + x)\cdot\sin(x - 90^\circ)} = -\dfrac{1}{\sin x}$

Solution

$$\text{LHS} \quad \frac{\sin^2 x + \cos^2 x}{(\tan x)(-\cos x)}$$

$$= \frac{1}{\left(\frac{\sin x}{\cancel{\cos x}}\right) \times \frac{(-\cancel{\cos} x)}{1}}$$

$$= \frac{1}{-\sin x} = \text{RHS}$$

Now do Activity 2 nos A and B.

More complementary angles

If $\alpha + \beta = 90^o$, we say α and β are complementary angles

we know $\sin \alpha = \cos\ (90^o\ ^\circ\ \alpha)$

so $\quad \sin 20^o = \cos 70^o$

$\quad \cos 40^o = \sin 50^o$

and $\quad \dfrac{\cos 10^\circ}{\sin 80^\circ} = 1$ and $\dfrac{\sin 20^\circ}{\cos 70^\circ} = 1$

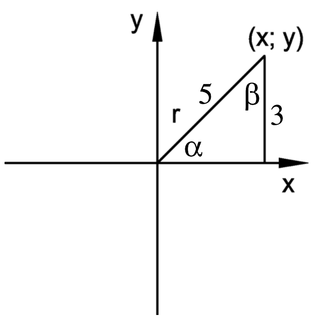
Example 5

If α and β are complementary angles and $\sin \alpha = \frac{3}{5}$, find $\tan \beta$.

Step 1:

$\sin \alpha = \frac{3}{5}$ But α and β are complementary angles so $\cos \beta = \frac{3}{5}$.

Step 2: Draw



$y = 3$

$r = 5$

By Pythagoras $x = 4$

So $\tan \beta = \frac{4}{3}$

Example 6

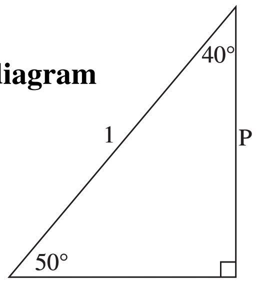
If $\sin 50^\circ = p$, find in terms of p

a) $\cos 40^\circ$ $\qquad$ b) $\cos 50^\circ$

Solution

- **Using a diagram**

sin 50° = p



According to Pythagoras $x = \sqrt{1 - p^2}$

So: (a) $\cos 40° = \frac{p}{1} = p$

(b) $\cos 50° = \frac{\sqrt{1-p^2}}{1}$

- **Using identities**

(a) $\cos 40° = \cos(90° - 50°)$

$= \sin 50°$

(b) cos 50°:

$\cos^2 50° + \sin^2 50° = 1$

$\therefore \cos^2 50° = 1 - \sin^2 50°$

$\therefore \cos 50 = \sqrt{1 - p^2}$

Now do Activity 3.

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

A. Simplify:

1) $1 - \frac{\sin^2(180° + A)}{1 - \cos(180° + A)}$

2) $\frac{\sin(360° - \theta)\sin(-180° - \theta)\cos(180° - \theta)}{\cos(-\theta)\tan(180° - \theta)\sin\theta}$

3) $\frac{1}{\tan(-\theta)\cos(180° - \theta)} - \frac{\cos^2(-\theta)}{\sin(180° - \theta)}$

4) $\frac{\sin^2(180° + \theta) + \cos^2(-\theta)}{\tan(360° - \theta)\cos(-\theta)}$

5) $\frac{\tan A}{\sin(180° - A)} + \frac{\sin^2(-A}{\cos(180° + A)}$

6) $\frac{\tan(180° + \theta) + \sin(-\theta)}{(1 + \cos(180° + \theta)}$

7) $\frac{\sin(180° - \theta)\sin\theta - \cos^2(360° - \theta)}{\tan(180° + \theta) + \frac{1}{\tan(-\theta)}}$

B. Prove the following identities:

1) $\frac{1}{\sin(180° - A) + 1} - \frac{1}{\sin A - 1} = \frac{2}{\cos^2 A}$

2) $2\cos(180° - x)\sin(-x) = \frac{2\tan(180° + x)}{1 + \tan^2(-x)}$

3) $\left(\frac{1}{\cos\theta} - \tan\theta\right)^2 = \frac{1 + \sin(360° - \theta)}{1 - \sin(\theta - 180°)}$

4) $\frac{1}{\cos(360° - \theta)} - \tan(180° - \theta) = \frac{\cos(360° - \theta)}{1 + \sin(-\theta)}$

Activity 2

A. Simplify

1) $\frac{\sin(90° - \theta)(360° - \theta)\cos(90° + \theta)}{\tan(-\theta)\cos(360° - \theta)\tan(180° + \theta)}$

2) $\frac{\cos(90° + \alpha)\cos(-\alpha)\sin(-\alpha)}{\sin(\alpha - 90°)\tan(360° - \alpha)\cos\alpha}$

3) cos(90° ° θ) tanθ –cos(°θ) ° cos(°θ)sin(θ ° 90°)

4) $\dfrac{\sin(180° - \beta)\tan(180° - \beta)\sin^2(90° - \beta)}{\cos(360° - \beta)\cos(\beta - 90°)\cos(-\beta)}$

5) $\dfrac{\tan(180° + \theta)\cos(90° - \theta)}{\sin(90° - \theta)} - \dfrac{\cos(180° - \theta)}{\sin(90° + \theta)}$

B. Prove that:

1) $\dfrac{\cos^2(90° - x) + \sin^2(90° + x)}{\tan(180° + x)\sin(x - 90°)} = \dfrac{1}{\sin x}$

2) $\left(\dfrac{\cos\alpha\cos(180° + \alpha)}{\cos(90° - \alpha)}\right) = \dfrac{1}{\tan^2(-\alpha)} - \sin^2(90° + \alpha)$

3) $\dfrac{\sin(-x)\tan(180° + x)\sin(90° + x)}{\cos(90° + x)\cos(-x)} = +\tan x$

4) $\dfrac{1}{\sin(180° - A) + 1} - \dfrac{1}{\cos(90° - A) - 1} = \dfrac{2}{\cos^2 A}$

5) Prove $\cos x(\tan x + \frac{1}{\tan x}) = \frac{1}{\sin x}$

Activity 3

1) Write the following at a ratio of 20°

a) sin 250° b) cos 340° c) tan 160°

2) If cos 35° = m, find in terms of m

a) sin 305° b) sin 245° c) cos 245°

3) If α and β are complementary angles and $\sin\alpha = \frac{4}{5}$, without a calculator find the value of $\sin\beta \circ \tan\alpha$

4) If x and y are complementary angles and $2\cos x = \sqrt{2}$, find

a) $\cos^2 x \circ \sin^2 y$ b) $\tan x \,.\, \tan y$

5) If A and B are supplementary angles and $\tan\beta = -\frac{8}{15}$, find sin A + cos B

6) If α and β are supplementary angles and $\cos\alpha = \frac{12}{13}$, find $\tan\beta \circ \sin\alpha$

7) If sin 25° = p, find in terms of p

a) cos 245° b) cos 155°

8) If tan 10° = m, find in terms of m

a) sin 10° b) cos 10° c) tan 80°

9) Without a calculator, prove

a) $\dfrac{2\cos 80°}{\sin 10°} = 2$ b) $\dfrac{\cos 340°}{2\sin 110°} = \dfrac{1}{2}$

c) $\dfrac{\sin 70° \cos 175°}{\cos 340° \cos 185°} = 1$ d) $\dfrac{\sin^2 70° + \sin^2 20°}{\sin^2 40°(1 + \tan^2 50° = 1)}$

TRIGONOMETRIC EQUATIONS

The general solution of trigonometric equations

Lessons 16 – 18

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
Determine the general solution of trigonometric equations.

Overview

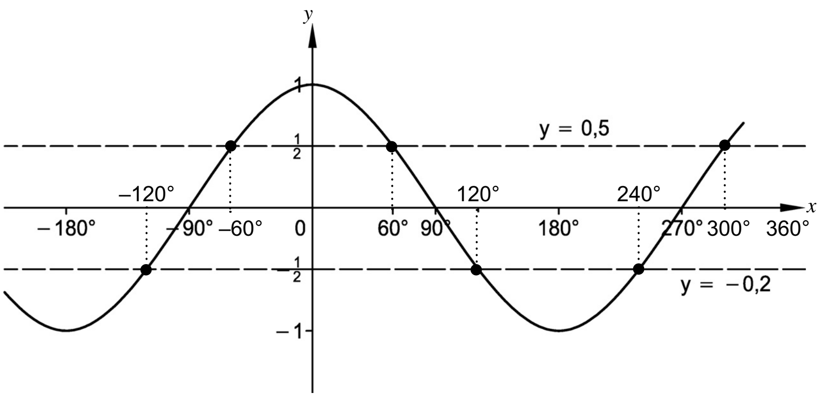
In this lesson you will:

- Establish the general solution for sin θ = p, cos θ = p and tan θ = p by looking at the trigonometric graphs studied in Grade 10.
- Use your calculator correctly to both establish trigonometrical ratios and angles.
- Solve trigonometric equations in a given domain.

Lesson

DVD

This is the graph of $y = \cos x$ which you learnt about in Grade 10.



Let's use it to solve cos θ = 0,5 (draw a line $y = \frac{1}{2}$ to see where the line intersects the graph).

As you can see there are many solutions. Those solutions repeat themselves every 360°, so we need to add multiples of 360°, which happens to be the period of the cos graph. So let's see what the calculator gives us.

Press [Shift cos 0,5] – you get 60° – now you have to use this to find all the other solutions.

From the graph the general solution is $\pm 60° + k \cdot 360°$; $k \in \mathbb{Z}$.

Let's now solve cos θ = –0,5 (Draw a line $y = -0{,}5$ to see where the line intersects the graph).

Once again there are many solutions. The calculator gives us 120°. To find all solutions we write

$\pm 120° + k \cdot 360°$; $k \in \mathbb{Z}$

General solutions for cos θ = p

Press [Shift cos p] to get θ and the general solution is $\pm \theta + k \cdot 360°$ $k \in \mathbb{Z}$

These solutions repeat themselves every 360°, so we need to add multiples of 360° (which is the period of the cosine graph)

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Example

1) Solve for θ is $2 \cos \theta = –0{,}646 \Rightarrow \cos \theta = 0{,}323$

Now: $\theta = \pm \cos^{-1} (–0{,}323) + k \cdot 360°$ $\therefore \theta = \pm 108{,}8 + k \cdot 360°$ $k \in \mathbb{Z}$

$\theta = \pm 108{,}8 + k \cdot 360°$ $k \in \mathbb{Z}$

2) Solve for θ if $\cos 3\theta = –0{,}632$. Get the angle $\rightarrow 3\theta = \pm \cos^{-1} (– 0{,}632) + k \cdot 360°$: $k \in \mathbb{Z}$

$\therefore 3\theta = \pm 129{,}2 + k \cdot 360°$.

We don't want 3θ we want θ, so $\theta = \pm 43{,}06 + k \cdot 120°$ $k \in \mathbb{Z}$. Notice here that the period of $\cos 3\theta$ is no longer $360°$, but $\frac{360°}{3} = 120°$

3) Solve for θ if $\cos(\theta – 40°) = \tan 22°$.

$\cos(\theta – 40°) = [0{,}4040262258]$

Leave that answer in the calculator and press shift cos to get the angle.

$\theta – 40° = \pm 66{,}2 + k \cdot 360°$ $\therefore \theta = 40° \pm 66{,}2° + k \cdot 360°$

$\theta = 106{,}2°$ or $–26{,}2°$

4) Solve for θ if $\cos (\theta – 50°) = \cos 2\theta$. You have the angle. Spread out

$\cos (\theta – 50°) = \cos 2\theta$

So: $\theta – 50° = \pm 2\theta + k \cdot 360°$

$\therefore \theta = 2\theta = 50° + k \cdot 360°$

Thus $3\theta = 50° + k \cdot 360°$ or $– \theta = 50° + k \cdot 360°$

$\theta = 16{,}7° + k \cdot 120°$ or $\theta = –50° + k \cdot 360°$ $k \in \mathbb{Z}$

5) Solve for θ if $\cos^2 2\theta = \frac{1}{4}$

$\cos 2\theta = \frac{\pm 1}{2}$

$2\theta = \pm \cos^{-1}\left(\frac{1}{2}\right) + k \cdot 360°$ or $2\theta = \pm \cos^{-1} \left(-\frac{1}{2}\right) + k \cdot 360°$

$2\theta = \pm 60° + k \cdot 360°$ or $2\theta = \pm 120° + k \cdot 360°$

$\theta = \pm 30° + k \cdot 180°$ or $\theta = \pm 60° + k \cdot 180°$ $k \in \mathbb{Z}$

6) $\cos 3\theta = – \cos \theta$

We need to remove the negative by reading from the left to the right.

$\cos 3\theta = – \cos \theta$ and is thus negative.

So $\cos 3\theta < 0$, and according to the CAST rule, this happens in the 2nd and 3rd quadrants

The horizontal reduction formulae in these quadrants are $180° – \theta$ and $180° + \theta$, so we can combine them to $(180° \pm \theta)$

So $– \cos \theta = \cos (180° \pm \theta)$

Now $\cos 3\theta = \cos (180° \pm \theta)$

$\therefore 3\theta = 180° \pm \theta + k \cdot 360°$

$\therefore 3\theta \pm \theta = 180° + k \cdot 360°$

If we split:

$3\theta \pm \theta = 180° + k \cdot 360°$	$3\theta – \theta = 180° + k \cdot 360°$
$4\theta + = 180° + k \cdot 360°$	$2\theta = 180° + k \cdot 360°$
$\theta + = 450° + k \cdot 90°$ … (1)	$\theta = 90° + k \cdot 120°$ … (2)

$k \in \mathbb{Z}$

Sine θ ∈ [– 180°; 90°]

For 1: θ = ± 45°; – 135°

From 2: θ = 90°; – 30°; –150°

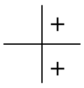
Co-ratio equations

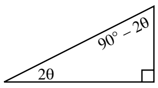
7. Solve for θ if cos (θ – 10°) = sin 2θ .

If this was cos (θ – 10°) = cos 2θ as in a similar example (4) above, then the ratios are already balanced. So we would only focus on the angles.

Here our duty is to get those ratios the same:

Remember sine becomes cosine in vertical reduction, so cos (θ – 10°) = cos 2θ and the sin 2 θ has a positive sign in front of it. So we ask where is (θ – 10°) positive?

According to the CAST rule  it will be positive in the first and 4th quadrants.

So in the first  sin 2θ = cos (90° – 2θ) and in the fourth quadrant sin 2 θ = cos (2θ – 90°)

So: cos (θ – 10°) = cos (90° – 2θ) or cos (θ – 10°) = cos (2θ – 90°)

$\therefore$ θ – 10° = 90° – 2θ + k·360° or θ – 10° = 2θ – 90° + k·360°

3θ = 100° + 360° –θ = –80° + k·360°

and θ = $\left(\frac{100}{3}\right)^{\circ}$ + k·120° $\therefore$ θ = 80° + k·360°; $k \in \mathbb{Z}$

Workbook: Lesson 16 Activity

For conclusion

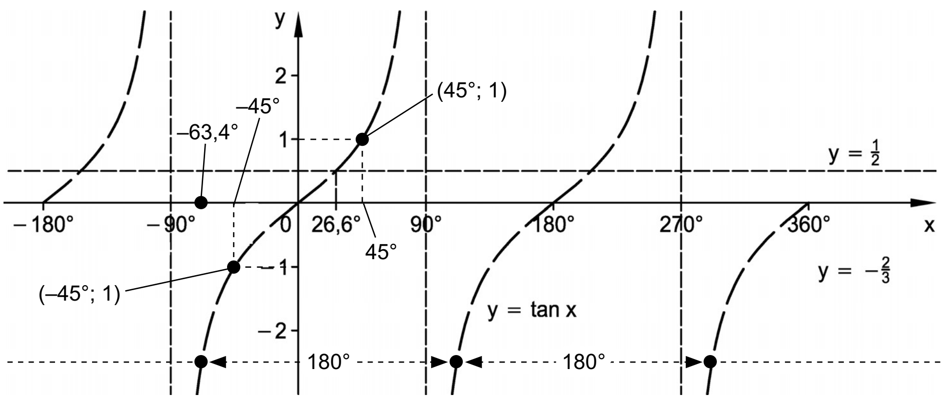
This is the graph of y = tan x (you learnt to draw it in Grade 10)

FORMATIVE ASSESSMENT

PAIRS

DVD

Lesson 17–18



We will use the graph to solve tan $x = \frac{1}{2}$

Draw the line $y = \frac{1}{2}$. (Press shift tan.5, and the calculator gives us 26,6°).

We need to include all the solutions. We simply add k·180° if $k \in \mathbb{Z}$ because the period of the tan graph is 180°

General solution: x = 26,6° + k·180° $k \in \mathbb{Z}$.

Now let's solve $\tan x = -\frac{2}{3}$ (Draw the line and see what the calculator gives you, then $+ k \cdot 180°$ $\quad k \in \mathbb{Z}$)

so $x = -63{,}4° + k \cdot 180° \quad k \in \mathbb{Z}$

General solution for tan $\theta = p$

$\therefore\ \theta = \tan^{-1}(p) + k \cdot 180° \quad k \in \mathbb{Z}$

$k \in \mathbb{Z}$

Example

1. Solve for θ

   $\tan 3\theta = -2{,}7$

   Press Shift tan (–2,7) to get the angle and write down the general solution.

   $3\theta = -69{,}7 + k \cdot 180°$ Divide by 3

   $\theta = -23{,}2 + k \cdot 60° \quad k \in \mathbb{Z}$

2. Solve for θ if $\theta \in [-180° ; 180°]$

   $\tan^2\theta = 0{,}81$

   $\tan\theta = \pm 0{,}9$ Split them

   $\tan\theta = 0{,}9$ or $\tan\theta = -0{,}9$ Find angles for the general solution

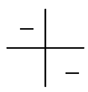
   $\theta = 42° + k \cdot 180°$ or $\theta = -42° + k \cdot 180° \quad k \in \mathbb{Z}$

   Substitute for k to find angles between $[-180° ; 180°]$

   $\{-138° ; -42° ; 42° ; 138°\}$

3. Solve for θ if $\tan 3\theta = -\tan(\theta + 20°)$

   $\tan 3\theta = -\tan(\theta + 20°)$

   $\tan 3\theta < 0$

   So $\tan 3\theta = \tan(180° - \theta - 20°)$

   $3\theta = 160° - \theta + k \cdot 180°$

   $\therefore\ 4\theta = 160° + k \cdot 180°$

   $\theta = 40° + k \cdot 45°, \quad k \in \mathbb{Z}$

   $\tan 3\theta = \tan(-\theta - 20°)$

   $3\theta = -\theta - 20° + k \cdot 180°$

   $\therefore\ 4\theta = -20° + k \cdot 180°$

   $\theta = -5° + k \cdot 45°$

   NB: Can you see that both answers are in fact the same. This is important, and that happens because of the period of the tan graph. So we only need to look at one quadrant.

4. Solve for x if $3\sin 2x = -2\cos 2x$

   $\frac{\sin 2x}{\cos 2x} = \frac{-2}{3}$ Trig identities

   $\tan 2x = \frac{-2}{3}$

   $2x = -33{,}7° + k \cdot 180°$

   $x = -16{,}85° + k \cdot 90°$

General solution for sin θ = p

$\sin\theta$ is +ve in quadrants 1 and 2

So if $\sin\theta = p$

then:
$\theta = \sin^{-1}(p)/180° - \sin^{-1}(p) + k\cdot 360°; \qquad k \in \mathbb{Z}$

Example 1

Find the general solution for θ in $\sin\theta = \frac{1}{2}$:

$\sin\theta = \frac{1}{2}$

$$\therefore\ \theta = \left\{\begin{matrix} \sin^{-1}\left(\frac{1}{2}\right) \\ 180° - \sin^{-1}\left(\frac{1}{2}\right) \end{matrix}\right\} + k\cdot 360°;$$

$$\therefore\ \theta = \left\{\begin{matrix} 30° \\ 150 \end{matrix}\right\} + k\cdot 360°, \quad k \in \mathbb{Z}$$

Example 2

$\sin 2\theta = -\frac{3}{4}$

$$\therefore\ 2\theta = \left\{\begin{matrix} \sin^{-1}\left(-\frac{3}{4}\right) \\ 180° - \sin^{-1}\left(-\frac{3}{4}\right) \end{matrix}\right\} + k\cdot 360°$$

$$\therefore\ 2\theta = \left\{\begin{matrix} -48{,}6° \\ 180°(-48{,}6°) \end{matrix}\right\} + k\cdot 360°$$

$$\therefore\ 2\theta = \left\{\begin{matrix} -48{,}6° \\ 228{,}6° \end{matrix}\right\} + k\cdot 360°$$

$$\therefore\ \theta = \left\{\begin{matrix} -24{,}3° \\ 114{,}3° \end{matrix}\right\} + k\cdot 180° \quad k \in \mathbb{Z}$$

Example

3. Solve for θ if $\sin 2\theta = -0{,}8$

$2\theta = -53{,}13° + k\cdot 360°$ or $2\theta = 180° - (-53{,}13°) + k\cdot 360°$

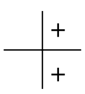
$\theta = -26{,}56° + k\cdot 180°$ or $2\theta = 233{,}13° + k\cdot 360°$

$\theta = 116{,}56° + k\cdot 180° \qquad k \in \mathbb{Z}$

Co-ratio equations

Solve for x if $\sin 2x = \cos 3x$

In front of the cos, there is a '+' sign

So $\sin 2x > 0$ and according to the CAST rule  , this is in the first and second quadrants.

So we need to make the cosine a sine, and we can only do this through vertical reduction. That is $\cos 3x$ becomes $\sin(90° \pm 3x)$

So: $\sin 2x = \sin(90° \pm 3x)$

$\therefore\ 2x = 90° \pm 3x + k\cdot 360°$

$\therefore 2x - 3x = 90° + k\cdot 360°$ or $2x + 3x = 90° + k\cdot 360°$

$-x = 90° + k\cdot 360°$ $\quad$ $5x = 90° + k\cdot 360°$

$\therefore x = -90° + k\cdot 360°$ $\quad$ $x = 18° + k\cdot 72°$; $\quad k \in \mathbb{Z}$

Trigonometric equations involving factorisation.

a) Common factor problems are those that usually have two terms.

Example

Solve for x: $3 \cos x \sin x = 2 \cos x$

Do not divide by $\cos x$ because you will lose solutions

$3 \cos x \sin x - 2 \cos x = 0$. $\quad$ Write in the form where one side is zero.

$\cos x\,(3 \sin x - 2) = 0$. $\quad$ Take out a common factor

$\cos x = 0$ $\quad$ or $\quad$ $\sin x = \frac{2}{3}$

$x = \pm 90° + k\cdot 360°$ $\quad$ or $\quad$ $x = 41{,}8° + k\cdot 360°$ $\quad$ or $\quad$ $x = 138{,}2° + k\cdot 360°$ $k \in \mathbb{Z}$

b) Trinomials (3 terms)

$a^2 - a - 2$ $\quad$ is a quadratic trinomial that we can factorise

so is $a^2 - ab - b^2$

In trigonometry we sometimes need to create trinomials.

Example

1. Solve for θ if $2 \sin^2\theta + 5 \cos \theta + 1 = 0$

   Look at the non–squared term.

   We need a trinomial in terms of $\cos \theta$ – that is with only cos and no sine terms.

   Use $\sin^2\theta = 1 - \cos^2\theta$

   $2(1 - \cos^2\theta) + 5 \cos \theta + 1 = 0$ $\quad$ Simplify

   $2 - 2 \cos^2\theta + 5 \cos \theta + 1 = 0$

   $-2 \cos^2\theta + 5 \cos \theta + 3 = 0$ $\quad$ Change signs

   $2 \cos^2\theta - 5 \cos \theta - 3 = 0$ $\quad$ Factorise $\rightarrow$ This is similar to $2x^2 - 5x - 3 = 0$ where $x = \cos \theta$.

   $(2\cos \theta + 1)(\cos \theta - 3) = 0$

   $\therefore\ 2 \cos \theta = -1$ $\quad$ or $\quad$ $\cos \theta = 3$

   $\therefore\ 2 \cos \theta = -1$ $\quad$ or $\quad$ $\underbrace{\cos \theta = 3}_{\text{invalid}}$

   $\therefore\ \cos \theta = -\frac{1}{2}$

   $\therefore\ \theta = \pm \cos^{-1}\left(-\frac{1}{2}\right) + k\cdot 360°$

   $= \pm 120° + k\cdot 360°$

   This is a quadratic trinomial with middle term $-3 \sin \theta \cos \theta$. The algebraic equivalent is: $a^2 + 2ab + b^2 = 0$. So we have $\cos^2\theta$, and $3 \sin \theta \cos \theta$ and we now need a $\sin^2 \theta$ to complete the trinomial. The only way we can bring that in is by using the 1, since $= \sin^2 \theta + \cos^2 \theta$.

So $\cos^2 \theta - 3 \sin \theta \cos \theta + \sin^2 \theta + \cos^2 \theta = 0$

2. Solve for θ if $\theta \in [-180° ; 360°]$ and

$\cos^2\theta - 3 \sin \theta \cos \theta + 1 = 0$

$\therefore \cos^2\theta - 3 \sin \theta \cos \theta + \sin^2\theta + \cos^2\theta = 0$

$\therefore 2 \cos^2\theta - 3 \sin \theta \cos \theta + \sin^2\theta = 0$

$\therefore (2 \cos \theta - \sin \theta)(\cos \theta - \sin \theta) = 0$ Factors

$\therefore 2 \cos \theta = \sin \theta$ or $\cos \theta = \sin \theta$ Create tan θ

$2 = \frac{\sin\theta}{\cos\theta}$ or $\frac{\sin\theta}{\cos\theta} = 1$

$\tan \theta = 2$ or $\tan \theta = 1$

$\theta = 63{,}4° + k \cdot 80°$ or $\theta = 45° + k \cdot 180°$ $k \in \mathbb{Z}$

$\{-116{,}6° ; -135° ; 45° ; 63{,}4° ; 225° ; 243{,}4°\}$

Equations with four terms: Group in two's

Example

Solve for x if $2 \sin^2 x + 2 \sin x \cos x + \sin x + \cos x = 0$

$2 \sin x (\sin x + \cos x) + (\sin x + \cos x) = 0$

$(\sin x + \cos x)(2 \sin x + 1) = 0$

$\sin x = -\cos x$ or $\sin x = -\frac{1}{2}$

$\tan x = -1$

$x = -45° + k \cdot 180°$ or $x = -30° + k \cdot 360°$ or $x = 210° + k \cdot 360,$ $k \in \mathbb{Z}$

Your Fact File

- If $\sin \theta = p$ and $-1 \leq p \leq 1$,
  then $\theta = \left\{\begin{matrix} \sin^{-1}(p) \\ 180° - \sin^{-1}(p) \end{matrix}\right\} + k \cdot 360°;$ $k \in \mathbb{Z}$
- If $\cos \theta = p$ and $-1 \leq p \leq 1$,
  then $\theta = \pm \cos^{-1}(p) + k \cdot 360°$; $k \in \mathbb{Z}$
- If $\tan \theta = p$ and $p \in \mathbb{R}$,
  then $\theta = \tan^{-1}(p) + k \cdot 180°$; $k \in \mathbb{Z}$
- $1 = \sin^2 \theta + \cos^2 \theta$
- If equations have 2 terms
  - You look for a common factor
  - You may have to use co-ratios
  - You may have to form the tan ratio if each side has a cos and sine of the same angle
- If equations have three terms, it is usually a quadratic trinomial
- If there are 4 terms, you have to group them in pairs

Activity 1

Solve for θ (if necessary correct to one decimal place)

1. $\cos 2\theta = -0{,}357$
2. $\cos(\theta + 50°) = -0{,}814$ if $\theta \in [-360° ; 360°]$
3. $\cos\theta = 0$
4. $\cos\theta = -1$
5. $\sin 3\theta = \cos(\theta + 62°)$ if $\theta \in [-180° ; 180°]$
6. $\cos 2\theta = \frac{1}{2}$
7. $\cos 2\theta = \sin(\theta - 40°)$

Activity 2

Solve for θ (if necessary correct to one decimal place)

1. $\tan 3\theta = \frac{-1}{\sqrt{3}}$
   $\theta \in [-60° ; 60°]$
2. $\tan(\theta + 20°) = \tan(130° - \theta)$
   $\theta \in [-180° ; 180°]$
3. $\tan\theta = -\tan 40°$
   $\theta \in [-360° ; 0°]$
4. $\sin\theta = 2\cos\theta$; $\theta \in [-180° ; 180°]$
5. $-\sin\theta = \frac{1}{2}\cos\theta$
   $\theta \in [-180° ; 180°]$
6. $\sin\theta + 3\cos\theta = 0$
   $\theta \in [-90° ; 180°]$

Activity 3

Instruction: Find the general solutions to the following equations.

1. $\sqrt{3}\tan\theta = -1$
2. $\sin 3\theta = \cos(\theta + 62°)$
3. $\cos 2\theta = -0{,}357$
4. $\sin 2\theta = -1$
5. $\sin(3\theta + 24°) = -0{,}279$
6. $\cos(\theta + 50°) = -0{,}814$
7. $\tan^2 3\theta = 3$
8. $\cos 2x = \sin(x - 40)$
9. $\tan(90° - x) = \tan(2x + 60°)$
10. $\sqrt{2}\cos\theta = -1$
11. $\sqrt{2}\sin 2\theta = -1$

Activity 4

In each case, solve the equation according to the given domain.

1. $2\sin^2\theta - 3\cos^2\theta = 2$ if $\theta \in [-360° ; 360°]$
2. $2\sin\theta\cos\theta - \sin\theta = 1 - 2\cos\theta$
3. $7\sin^2\theta + 4\sin\theta\cos\theta = 3\theta \in [-180° ; 360°]$
4. $\cos(5\theta - 40° = \sin(2\theta - 20°)$
5. $8\cos x = 4\sin^2 x - 7$ if $x \in [-360° ; 0°]$
6. $5\sin^2\theta + 2\cos\theta - 5\cos\theta\sin\theta = 2\sin\theta$
7. $\sin 2A = \tan A$ if $A \in [-180° ; 180°]$
8. $4\sin^2 x + 2\cos^2 x = 5\sin x$ if $x \in [-180° ; 360°]$

Activity 5

Solve:

1. $\cos 2A = -\cos 3A$
2. $2\cos^2 x = 3\sin x + 3$ if $x \in [-360° ; 360°]$
3. $10\cos^2\theta - 10\sin\theta\cos\theta + 2 = 0$
4. $\cos(\theta - 30°) = \sin 50°$
5. $\cos 3x = -\sin 4x$
6. $3\sin^2 x - 5\cos^2 x = 0$
7. $\cos^2 x - \sin^2 x = 3\sin x + 2$
8. $\cos^2\theta - 1 = 2\sin\theta\cos\theta$
9. $2\sin x - \frac{\cos x}{\sin x} + 2\cos x = 1$ if $x \in [-180° ; 360°]$

QUADRATIC SEQUENCES (1)

Lesson 19

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
Assessment Standards
Investigate number patterns (including but not limited to those where there is a constant second difference between consecutive terms in a number pattern, and the general term is therefore quadratic) and hence:
(a) Make conjectures and generalization.
(b) Provide explanations and justifications and attempt to prove conjectures.

Overview

In this lesson you will:

- Revise grade 10 first order difference patterns.
- Study second order difference patterns

Lesson

LC DVD

Revision of linear number patterns

Consider the linear number pattern 3; 5; 7; 9; 11; …

The **first term** (T_1) is 3.

The pattern is formed by adding 2 to each new term.

We say that the **constant difference** between the terms is 2.

$T_1 = 3$

$T_2 = 3 + 2$

$T_3 = 3 + 2 + 2 = 3 + (2) \times 2$

$T_4 = 3 + 2 + 2 + 2 = 3 + (3) \times 2$

$T_6 = 3 + 2 + 2 + 2 + 2 + 2 = 3 + (5) \times 2$

$T_7 = 3 + 2 + 2 + 2 + 2 + 2 + 2 = 3 + (6) \times 2$

$T_{50} = 3 + (49) \times 2 = 101$

$\therefore T_n = 3 + (n - 1)2$

You will probably notice that the first term in this formula is 3 and the constant difference is 2.

In general, these linear number patterns have the general form:

$\therefore T_n = a + (n - 1)d$

where a represents the first term and d represents the constant difference.

In the previous example, the general term will be

$\therefore T_n = 3 + (n - 1)2$

$\therefore T_n = 3 + 2n - 2$

$\therefore T_n = 2n + 1$

Further example

Consider the number pattern: 4; 9; 14; 19; ……..

Determine the nth term and hence the 100^{th} term.

You will probably notice that the first term in this formula is 4 and the constant difference is 5.

$\therefore T_n = 4 + (n - 1)(5)$

$\therefore T_n = 4 + 5n - 5$

$\therefore T_n = 5n - 1$

$\therefore T_{100} = 5(100) - 1$

$\therefore T_{100} = 499$

Quadratic number patterns

We will now focus on **quadratic number patterns** with general terms of the form

$T_n = an^2 + bn + c$.

Consider the pattern 2; 5; 10; 17; 26; ………

The general term of the pattern is $T_n = n^2 + 1$

This general term works since:

$T_1 = (1)^2 + 1 = 2$

$T_2 = (2)^2 + 1 = 5$

$T_3 = (3)^2 + 1 = 10$

$T_4 = (4)^2 + 1 = 17$

$T_5 = (5)^2 + 1 = 26$

But the question arises as to how one actually gets the general term.

The following method with assist you in this regard.

2 5 10 17 26

3 5 7 9 → First difference

2 2 2 → Second difference

It is clear that this number pattern does not have a constant first difference. However, there is a constant second difference.

Suppose that the general term of a particular quadratic number pattern is given by $T_n = an^2 + bn + c$.

The terms of the number pattern would then be:

$T_1 = a(1)^2 + b(1) + c = a + b + c$ $\quad$ $T_2 = a(2)^2 + b(2) + c = 4a + 2b + c$

$T_3 = a(3)^2 + b(3) + c = 9a + 3b + c$ $\quad$ $T_4 = a(4)^2 + b(4) + c = 16a + 4b + c$.

$a + b + c$ $\quad$ $4a + 2b + c$ $\quad$ $9a + 3b + c$ $\quad$ $16a + 4b + c$

$3a + b$ $\quad$ $5a + b$ $\quad$ $7a + b$ → First difference

$2a$ $\quad$ $2a$ → Second difference

You will notice that the constant second difference is given by the expression $2a$.

The first term in the first difference row is given by $3a + b$ and the first term is given by $a + b + c$.

So, consider the previous number pattern: 2; 5; 10; 17; 26; ……..

$$\begin{array}{lccccccccc}
a + b + c & 2 & & 5 & & 10 & & 17 & & 26 \\
3a + b & & 3 & & 5 & & 7 & & 9 & \longrightarrow \text{First difference} \\
2a & & & 2 & & 2 & & 2 & & \longrightarrow \text{Second difference}
\end{array}$$

$2a = 2$	$3a + b = 3$	$a + b + c = 2$
$\therefore a = 1$	$\therefore 3(1) + b = 3$	$\therefore 1 + 0 + c = 2$
	$\therefore b = 0$	$\therefore c = 1$

Therefore the general term is $T_n = 1n^2 + 0n + 1 = n^2 + 1$

Further example

Consider the number pattern: 9; 16; 28; 45; 67; ……..

(a) Determine the general term.

(b) Determine the 40^{th} term.

(a)

$$\begin{array}{lccccccccc}
a + b + c & 9 & & 16 & & 28 & & 45 & & 67 \\
3a + b & & 7 & & 12 & & 17 & & 22 & \\
2a & & & 5 & & 5 & & 5 & &
\end{array}$$

$2a = 5$	$3a + b = 7$	$a + b + c = 9$
$\therefore a = \frac{5}{2}$	$\therefore 3\left(\frac{5}{2}\right) + b = 7$	$\therefore \frac{5}{2} - \frac{1}{2} + c = 9$
	$\therefore b = 7 - \frac{15}{2} = -\frac{1}{2}$	$\therefore c = 7$

Therefore the general term is $T_n = \frac{5}{2}n^2 - \frac{1}{2}n + 7$

(b) $T_n = \frac{5}{2}n^2 - \frac{1}{2}n + 7$

$\therefore T_{40} = \frac{5}{2}(40)^2 - \frac{1}{2}(40) + 7$

$\therefore T_{40} = 3\ 987$

LC INDIVIDUAL

LC SUMMATIVE ASSESSMENT

Activity 1

For each of the following number patterns determine the next two terms, the general term and hence the 150th term.

(a) 4; 7; 10; 13; ……………………………..

(b) 0; 4; 8; 12; ………………………………

(c) 10; 6; 2; ………………………………..

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. For each of the following number patterns determine the general term.

   (a) 3; 12; 27; 48; ……………………………..

   (b) 7; 24; 51; 88; ……………………………

   (c) 0; 3; 8; 15; ………………………………..

   (d) 6; 17; 34; 57; …………………….

3. Consider the following number pattern:

   $1 = 1$

   $1 + 2 = 3$

   $1 + 2 + 3 = 6$

   $1 + 2 + 3 + 4 = 10$

   $1 + 2 + 3 + 4 + 5 = 15$

   (a) Determine a general rule in terms of n for evaluating:

   $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n$

   (b) Hence calculate the value of:

   $1 + 2 + 3 + 4 + 5 + 6 + 7 + \ldots + 200$

Lesson 20

QUADRATIC SEQUENCES (2)

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
Assessment Standards
Investigate number patterns (including but not limited to those where there is a constant second difference between consecutive terms in a number pattern, and the general term is therefore quadratic) and hence:
(a) Make conjectures and generalization.
(b) Provide explanations and justifications and attempt to prove conjectures.

Overview

In this lesson you will:

- Revise second order difference patterns..
- Study recursive number patterns.

Lesson

Revision of quadratic number patterns

Quadratic number patterns have the general term $T_n = an^2 + bn + c$.

$a + b + c$ = first term

$3a + b$ = first term of first difference row

$2a$ = second difference

Example

Determine the general term of the number pattern 1; 5; 13; 25; 41; …………

$$\begin{array}{ccccccccc} 1 & & 5 & & 13 & & 25 & & 41 \\ & 4 & & 8 & & 12 & & 16 & \\ & & 4 & & 4 & & 4 & & \end{array}$$

It is clearly a quadratic number pattern because it has a constant second difference.

You can now proceed as follows:

$2a = 4$	$3a + b = 4$	$a + b + c = 1$
$\therefore a = 2$	$\therefore 3(2) + b = 4$	$\therefore 2 - 2 + c = 1$
	$\therefore b = -2$	$\therefore c = 1$

Therefore the general term is $T_n = 2n^2 - 2n + 1$

Recursive linear number patterns

Consider the number pattern: 3; 5; 7; 9; 11; ………….

$T_1 = 3$

$T_2 = 3 + 2 = T_1 + 2$

$T_3 = 5 + 2 = T_2 + 2$

$T_4 = 7 + 2 = T_3 + 2$

$T_5 = 9 + 2 = T_4 + 2$

$T_6 = 11 + 2 = T_5 + 2$

$\therefore T_{n+1} = T_n + 2$

This formula represents the **recursive form** of the number pattern.

Further examples

(a) Express in recursive form:

4; 10; 16; 22; 28; 34; ………..

(b) Determine the number pattern if:

$T_1 = 2$ and $T_n = T_{n-1} + 5$

(a) 4; 10; 16; 22; 28; 34; ………..

$T_1 = 4$

$T_2 = 4 + 6 = T_1 + 6$

$T_3 = 10 + 6 = T_2 + 6$

$T_4 = 16 + 6 = T_3 + 6$

$\therefore T_n = T_{n-1} + 6$

(b) $T_1 = 2$

$T_2 = T_1 + 5 = 2 + 5 = 7$

$T_3 = T_2 + 5 = 7 + 5 = 12$

$T_4 = T_3 + 5 = 12 + 5 = 17$

Therefore the number pattern is:

2; 7; 12; 17; 22; …………………..

Recursive quadratic number patterns

Example

Express the following number patterns in recursive form:

(a) 2; 5; 10; 17; 26; ………..

(b) 9; 16; 28; 45; 67; ………

(a) $T_1 = 2$

$\therefore T_2 = 2 + 3 = T_1 + 3$

$\therefore T_3 = 5 + 5 = T_2 + 5$

$\therefore T_4 = 10 + 7 = T_3 + 7$

$\therefore T_5 = 17 + 9 = T_4 + 9$

$\therefore T_n = T_{n-1} +$??????

Now consider the pattern **2**; 3; 5; 7; 9; …..

The first term **2** is the odd man out in the pattern because 3; 5; 7; 9; … is linear with a constant difference of 2 between the terms. With the first term, 2, you have to add 1 to get to the second term, 3.

What you can do is work out the general term for the pattern 3; 5; 7; 9; … and then subtract the odd man out.

For the linear pattern 3; 5; 7; 9; ……

$T_n = 3 + (n - 1)(2)$

$\therefore T_n = 3 + 2n - 2$

$\therefore T_n = 2n + 1$

Now let's remove the odd man out by subtracting 1 from n:

$T_n = 2(n - 1) + 1$

$\therefore T_n = 2n - 2 + 1$

$\therefore T_n = 2n - 1$

If we now add in $(2n - 1)$ to the recursive formula, we will get:

$T_n = T_{n-1} + (2n - 1)$

We can now verify this recursive formula as follows:

$T_1 = 2$

$T_2 = T_1 + (2(2) - 1) = 2 + 3 = 5$

$T_3 = T_2 + (2(3) - 1) = 5 + 5 = 10$

$T_4 = T_3 + (2(4) - 1) = 10 + 7 = 17$

Clearly, the recursive formula generates the terms of the number pattern.

(b) 9; 16; 28; 45; 67; ………

$T_1 = 9$

$\therefore T_2 = 9 + 7 = T_1 + 7$

$\therefore T_3 = 16 + 12 = T_2 + 12$

$\therefore T_4 = 28 + 17 = T_3 + 17$

$\therefore T_5 = 45 + 22 = T_4 + 22$

$\therefore T_n = T_{n-1} +$??????

Now consider the pattern **9**; 7; 12; 17; 22; …..

The first term **9** is the odd man out in the pattern because is linear with a constant difference of 5 between the terms. With the first term, 9, you have to subtract 2 to get to the second term, 7.

What you can do is work out the general term for the pattern 7; 12; 17; 22; …..

and then subtract the odd man out.

For the linear pattern 7; 12; 17; 22; …..

$T_n = 7 + (n - 1)(5)$

$\therefore T_n = 7 + 5n - 5$

$\therefore T_n = 5n + 2$

Now let's remove the odd man out by subtracting 1 from n:

$T_n = 5(n - 1) + 2$

$\therefore T_n = 5n - 5 + 2$

$\therefore T_n = 5n - 3$

If we now add in $(5n - 3)$ to the recursive formula, we will get:

$T_n = T_{n-1} + (5n - 3)$

We can now verify this recursive formula as follows:

$T_1 = 9$

$T_2 = T_1 + (5(2) - 3) = 9 + 7 = 16$

$T_3 = T_2 + (5(3) - 3) = 16 + 12 = 28$

$T_4 = T_3 + (5(4) - 3) = 28 + 17 = 45$

Clearly, the recursive formula generates the terms of the number pattern.

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. Write each of the following number patterns in recursive form.
   (a) 4; 7; 10; 13; ……………………………..
   (b) 0; 4; 8; 12; ………………………………
   (c) 10; 6; 2; ………………………………..
2. Determine the number pattern in each of the following cases:
   (a) $T_1 = 3$ and $T_n = T_{n-1} + 3$
   (b) $T_1 = 2$ and $T_n = 3T_{n-1}$

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

Write each of the following number patterns in recursive form.

(a) 7; 24; 51; 88; ……………………………..

(b) 3; 12; 27; 48; ……………………………..

(c) 6; 17; 34; 57; ……………………………..

Lesson 21

DIFFERENT SEQUENCES

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and number relationships
Assessment Standard

- Investigate number patterns including but not limited to those where there is a constant difference and hence.
- Make conjectures and generalisations,
- Provide explanations and justifications and attempt to prove conjectures.

Overview

In this lesson you will:

- Look at unusual sequences where there is no first or second difference.
- If possible find the nth term.
- Prove conjectures.

Lesson

Example 1

What will the next three terms be in the sequence 2; 6; 18; 54; …

	T_1	T_2	T_3	T_4		
$\Rightarrow$	2 ;	6 ;	18 ;	54 ;	…	
	$\frac{1}{3} \times 6$;	1×6 ;	3×6 ;	9×6 ;		
=	$3^{-1} \times 6$;	$3^0 \times 6$;	$3^1 \times 6$;	$3^2 \times 6$;	$3^3 \times 6$;	$3^4 \times 6$

So the next 3 terms will be: 162; 486; 1458

and $T_n = 3^{n-2} \times 6 = 3^n.3^{-2}.3.2 = 2.3^{n-1}$

Alternatively:

	T_1	T_2	T_3	T_4		
$\Rightarrow$	2 ;	6 ;	18 ;	54;		
=	2×1 ;	2×3 ;	2×9 ;	2×27		
=	2×3^0 ;	2×3^1 ;	2×3^2 ;	2×3^3 ;	2×3^4 ;	2×3^5 ;

…

So the next 3 terms will be: $2 \times 3^4 = 162$

$2 \times 3^5 = 486$

$2 \times 3^6 = 1458$

Example 2

Again they all are multiples of 6:

	T_1	T_2	T_3			
$\Rightarrow$	24 ;	12 ;	6			
=	4×6 ;	2×6 ;	1×6 ;	$\frac{1}{2} \times 6$;	$\frac{1}{4} \times 6$;	$\frac{1}{8} \times 6$
=	$2^2 \times 6$;	$2^1 \times 6$;	$2^0 \times 6$;	$2^{-1} \times 6$;	$2^{-2} \times 6$;	$2^{-3} \times 6$
=	$2^3 \times 6$;	$2^2 \times 3$;	$2^1 \times 3$;	$2^0 \times 3$;	$2^{-1} \times 6$;	$2^{-2} \times 3$

Next 3 terms will be: $3; \frac{3}{2}; \frac{3}{4}$

$T_n = 2^{4-n}.3$

Alternatively

24; 12; 6; …

$24; \frac{24}{2}; \frac{24}{4}; \frac{24}{8}; \frac{14}{16}; \frac{24}{32}; \ldots$

$Tn = 24. \frac{1}{2}^{n-1}$

Both these general terms are the same:

$T_n = 2^{4-n}.3$

and $T_n = 24. (\frac{1}{2})^{n-1}$

$= 8.3.2^{1-n}$

$= 2^3.3.2^{1-n}$

$= 3.2^{4-n}$

Example 3 (Problem solving)

A rubber ball is dropped from a height of 30m. After each bounce, it returns to a height that is 4/5 of the previous height.

a) express the first three heights as a sequence

Solution

$30\left(\frac{4}{5}\right); 30\left(\frac{4}{5}\right)^2, 30\left(\frac{4}{5}\right)^3$

b) how high will the ball be after 21 bounces?

Solution

$T_{21} = 30\left(\frac{4}{5}\right)^{21}$

Activity 1

Example 4 (The Fibonacci sequence)

FORMATIVE ASSESSMENT

PAIRS

DVD

Write down the next three terms in the sequence 1; 1; 2; 3; 5; 8;

Solution

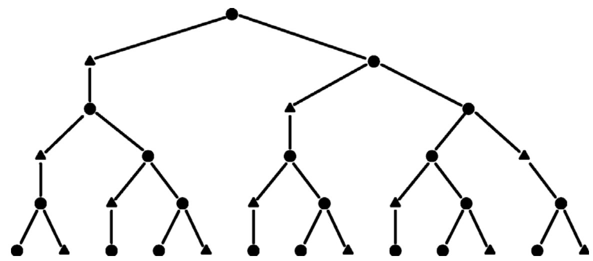
13; 21; 34

These are fascinating numbers because they appear all over in our world.

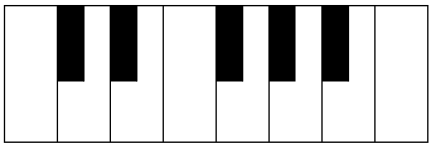
THE FAMILY

(In this family tree a male is represented by the symbol (▲) and a female by the symbol ●)

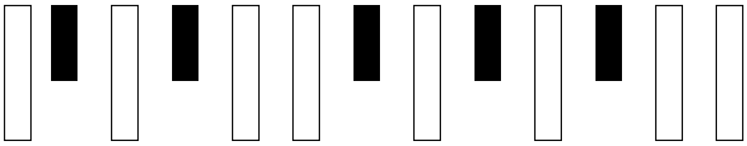
Generation back	Number of bees in each generation
1st	1
2nd	2
3rd	3
4th	5
5th	8
6th	13



The 13 keys shown below form one octave of a piano.



If we spread them out



What do we notice?

5 black keys

8 white keys

13 keys

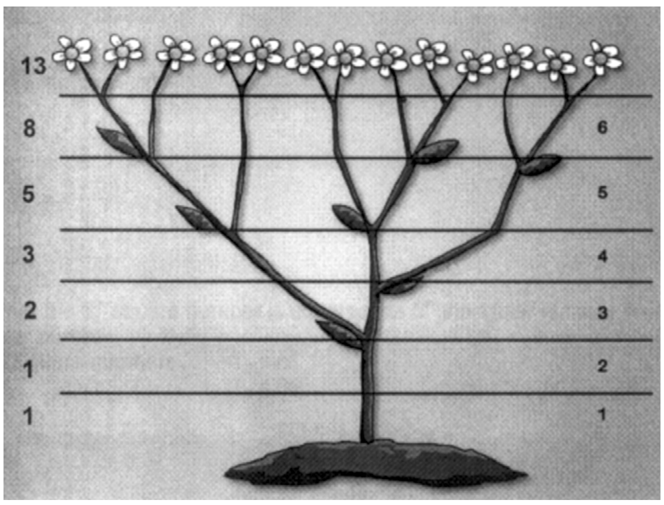
ALL Fibonacci numbers

In the 6th generation back of the male bee we have

● ▲ ● ▲ ● ● ▲ ● ▲ ● ▲ ● ●

Males are the black keys and females the white keys.

Look at this example from the internet.



www.mcs.surrey.ac.uk/personal/R.Knott/Fibonacci/Fibonat.html

The golden ratio

1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; 144

$\frac{T2}{T1} = 1$; $\frac{T3}{T2T} = T2$; $\frac{T4}{T3T} = T1,5$; $\frac{T5}{T4} = 1,666$; $\frac{T6}{T5T} = T1,6$

$\frac{13}{8} = 1,625$; $\frac{21}{13} = 1,61538\ldots$; $\frac{34}{21} = 1,61904$; $\frac{55}{34} = 1,6176\ldots$

$\frac{89}{55} = 1,618618\ldots$; $\frac{144}{89} = 1,617977\ldots$

Notice, as the sequence gets larger so the ratio gets closer to 1,62.

We call this the GOLDEN RATIO.

Workbook: Lesson 21

Activity 2

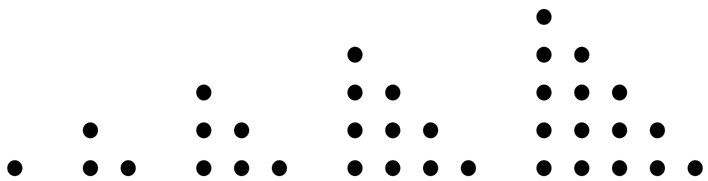
PORTFOLIO

INDIVIDUAL

DVD

Triangular numbers

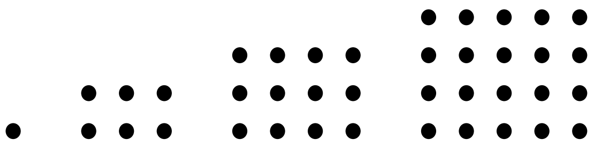
Numbers that form triangles



1 3 6 10 15

Look at the picture and find the rule.

Lets make rectangles



$T_1 = 1$

$T_2 = 2 \times \frac{3}{2}$

$T_3 = 3 \times \frac{4}{2}$

$T_4 = 4 \times \frac{5}{2}$

$T_n = \frac{n(n+1)}{2}$

Show the 6th square number is equal to the 5th triangular number plus the 6th triangular number.

Square numbers: $T_n = n^2$

$T_6 = 36$

Triangular numbers $T_n = \frac{n(n+1)}{2}$

$T_5 = 15$ $\quad$ $T_6 = 21$

$21 + 15 = 36$

Try more examples

Alternate exploration:

1 3 6 10 15

2 3 4 5

1 1 1

	T_1	T_2	T_3	T_4
$\frac{1}{2}n^2$	$\frac{1}{2}$	2	$\frac{9}{2}$	8
want	1	3	6	10
snt	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$

So $Tn = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n}{2}(n+1)$

Make a conjecture

The nth square is equal to the some of the nth and the $(n - 1)^{th}$ triangular numbers.

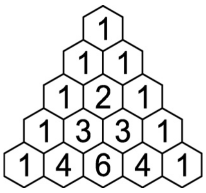
Now prove it: Formula for the triangular numbers

$\frac{n(n + 1)}{2}$ so $\frac{n(n + 1)}{2} + \frac{(n - 1)(n)}{2} = \frac{n^2 + n + n^2 - n}{2}$

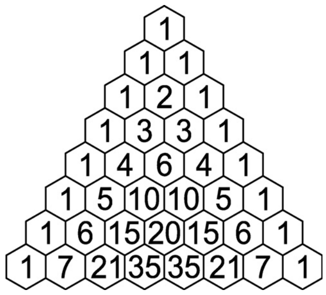
$= \frac{2n^2}{2}$

$= n^2$

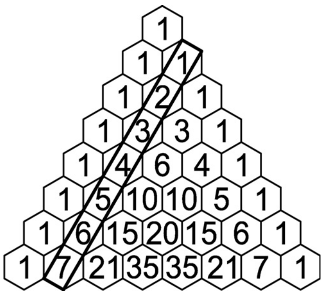
PASCAL'S TRIANGLE



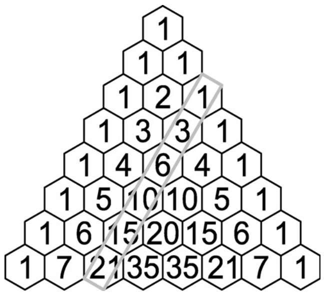
Complete the next three lines



a) Look for a linear sequence



b) Look for triangular numbers



c) Find the sum of the terms in the 51^{st} row

Row	Sum	Gen
1	1	20
2	2	21
3	4	22
4	8	23
5	16	24
6	32	25

1^{st} Row 1 2^{nd} Row 2 3^{rd} Row 4 4th Row 8

5th Row 2^4 6th Row 2^5 nth Row 2^{n-1} 51^{st} Row = 2^{50}

So $T_n = 2^{n-1}$

$T_{51} = 2^{50}$

Activity 1

PORTFOLIO
INDIVIDUAL
DVD

1. Find the nth term of the following sequences.
   a) $4; -2, 1; -\frac{1}{2}; \frac{1}{4}; \ldots$
   b) $-\frac{1}{8}; -\frac{1}{2}; -2; -8; \ldots$
   c) $32; 16; 8; 4; 2; \ldots$
   d) $3a; 6a^2; 9a^3; 18a^4 \ldots$
2. a) Find T_n
   b) Find the 8th term $9; 3;1 \ldots$
3. Which term of the sequence
   $1; \frac{3}{2}; \frac{9}{4} \ldots$ is equal to $\frac{243}{32}$? (Hint; Find T_n first)
4. A tree grows 120 cm during the first year. Each year it grows $\frac{9}{10}$ of the previous year's growth. How much in the 58th year?

Activity 2

PORTFOLIO

Write a paragraph on Leonardo Fibonacci.

Find out all you can about the Fibonacci numbers and where we find them in nature (you may look at various flowers, spirals, shells, the pineapple).

Generate the Fibonacci sequence by looking at the reproduction of rabbits.

Investigate the Golden Ratio and why it represents a "mathematically ideal" ratio.

Investigate the history of the golden ratio, including its role in the ancient Greek architecture.

Learn how to construct the Golden Rectangle.

You must supply a reference to the sources you used including web addresses.

Activity 3

1. a) Show why the 5th triangular number is $T_5 = \frac{1}{2} \times 5 \times 6$
   b) Find an expression in n for the nth triangular number.
2. a) Use the diagrams of the 3rd and 4th triangular numbers, T_3 and T_4 to show that their sum that their sum is the 4th square number, i.e. $S_4 = T_3 + T_4$
   b) Generalise with a formula the connection between triangular and square numbers.
3. a) Write down a table of square numbers from the 1st to the tenth.
   b) Find two square numbers which add to give a square number.
   c) Repeat part (b) for at least three other pairs of square numbers.

4. Without using a calculator explain whether:

a) 441 b) 2001

c) 1007 d) 4096 is a square number

5. Show that the difference between any two consecutive square numbers is an odd number.

6. Show that the difference between

a) The 7th square number and the 4th square number is a multiple of 3.

b) The 8th square number and the 5th square number is a multiple of 3.

c) The 11th square number and the 7th square number is a multiple of 4.

d) Generalise the statement implied in parts (a) (b) and (c).

e) 64 is equal to the 8th square number $S_8 = 8^2$

64 is equal to the 4th cube number $C_4 = 4^3$

Find other cube numbers which are also square numbers.

If you can, make a general comment about such cube numbers.

7. For any positive whole number n its “Tan function” $t(n)$ is defined as the number of positive whole number factors of n.

7 is a prime number. It has two factors so $t(7) = 2$

a) Show that if p is any prime number the $t(p) = 2$

b) For any prime number p and any positive value of n, find an expression for $t(p^n)$.

c) $6 \times 7 = 42$

The factors of 42 are:

1, 2, 3, 6, 7, 14, 21, 42 so $t(42) = 8$ and $t(6) = 4$

$t(7) = 2$

$\therefore\ t(6) \times t(7)\ n = 8$ so $t(6 \times 7) = t(6) \times t(7)$

Investigate to see whether $t(n \times m) = t(n) \times t(m)$

8. One way of making the number 5 by adding ones and threes is:

$5 = 3 + 1 + 1$ … and another different way is: $5 = 1 + 3 + 1$

Investigate the number of different ways of making any number by adding ones and threes.

9.

$1^3 + 2^3$	$1^3 + 2^3 + 3^3 + 4^3$	$1^3 + 2^3 + 3^3$
$= 1 + 8$	$= 1 + 8 + 27 + 64$	$= 1 + 8 + 27$
$= 9$	$= 100$	$= 36$
$= 3^2$	$= 10^2$	$= 6^2$
$= (1 + 2)^3$	$= (1 + 2 + 3 + 4)^2$	$= (1 + 2 + 3)^2$

Investigate this situation further. Try other powers.

10. The last digit of 146 is 6. this is written LD(146) = 6. What comments can you make about

    a) LD($n \times m$)

    b) LD(any square number)

    c) Show that $10n + 7$ can never be a square number for any positive whole number value of n.

11. Mystery sequences

    Find the next three terms of the following sequences:

    6; 8; 12; 14; 18 …

    767; 294; 72; …

    1; 5; 12; 22 … and find T_n

Lesson 22

SOLUTIONS OF TRIANGLES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape, space and measurement
Assessment Standard
Solutions of triangles using the sine, cosine and area rule.

Overview

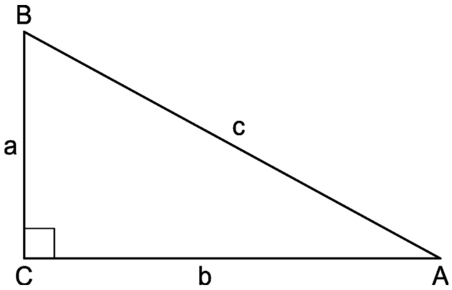
In this lesson you will:

- Review what you learnt in Grade 10.
- Prove the area rule, and the sine rule.
- Apply the sine and area rule to real life problems.

DVD

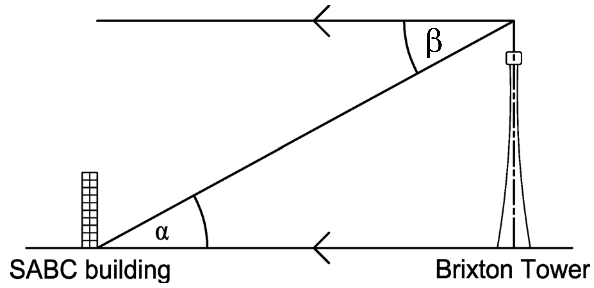
Lesson

Grade 10 Work



For angle A:	For angle B:
a is the opposite side	a is the adjacent side
b is the adjacent side	b is the opposite side
c is the hypotenuse	c is the hypotenuse

Angles of elevation and depression.



Angle of elevation from the bottom of the SABC building to the top of the tower is α.

The angle of depression from the top of the tower to the bottom of the SABC building is β. Why is $\alpha = \beta$?

The area of a triangle is $\frac{1}{2}$ base x height. What do we do if we do not have the height?

Yes! We use trigonometry.

Example 1

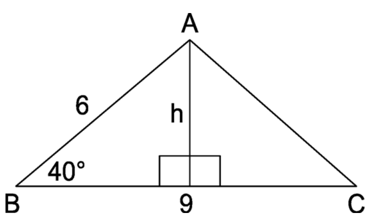
Find the area of $\triangle ABC$

Let's draw in height

Area = $\frac{1}{2}$ base x height

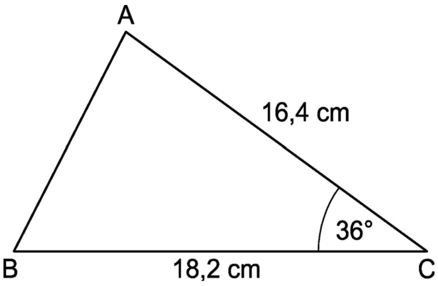
base = 9 $\qquad \frac{h}{6} = \sin 40°$

$\therefore$ h = 6 sin 40°



Area = $\frac{1}{2} \times 9 \times 6 \sin 40° = 17{,}36 \text{ u}^2$

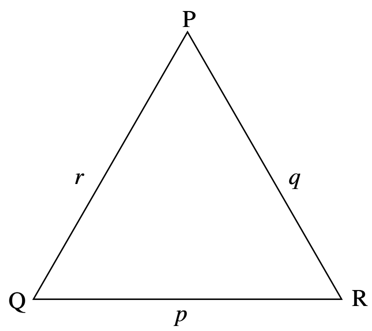
We do not need to find the height every time.



If the information is grouped so that I have two sides and the angle included by the two sides, then I can apply the area rule.

Area = $\frac{1}{2} \times 16{,}4 \times 18{,}2 \sin 36°$

$= 87{,}7 \text{ cm}^2$



Area $\triangle$PQR $= \frac{1}{2} pq \sin \text{R}$

$= \frac{1}{2} rq \sin \text{P}$

$= \frac{1}{2} pr \sin \text{Q}$

Let's prove the area rule

We use the co-ordinate system.

To find the height of the triangle, we need to find the co-ordinates of point R:

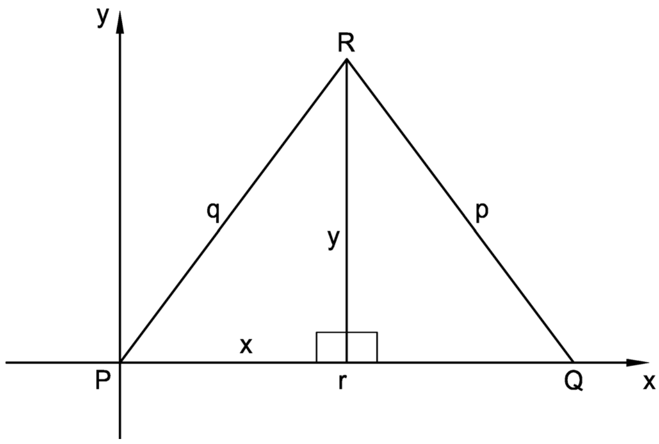
$\frac{x}{q} = \cos \text{P}$

$x = q \cos \text{P}$

$\frac{y}{q} = \sin \text{P}$

$y = q \sin \text{P}$

So R($q$cos P; $q$sin P)



Area $\triangle$PQR $= \frac{1}{2}$ base $\times$ height

$= \frac{1}{2}$ r $\times$ y – co-ordinate at R

$= \frac{1}{2}$ rq sin P

Similary if R were at the origin

Area $\triangle$PQR $= \frac{1}{2}$ pq sin R

And if Q were at the origin

Area $\triangle$PQR $= \frac{1}{2}$ pr sin Q

Example 2

The base of the prism is a triangle and not the rectangle on which it stands.

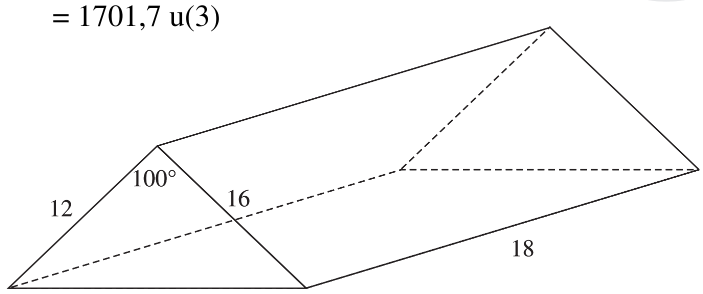
Remember: the side we choose as our base has to be the part that is uniform.

Find the volume of this triangular prism

Volume = Area of base x height

= ½ x 12 x 16 sin 100° x 18

= 1701,7 u(3)

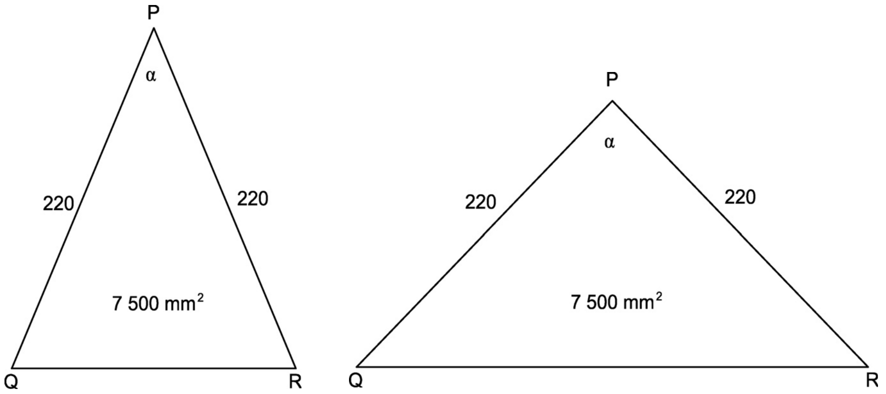


Example 3

The equal sides of an isosceles triangle are 220mm. Find the possible sizes of the angles of the triangle if the area is 7500mm^2.

Let's draw the triangle.

Whoops, there are two possible triangles.



Area $= \frac{1}{2}(220)(220) \sin \alpha$

7500 $= (0{,}50(220)(220) \sin a$

$\sin \alpha = \frac{7500}{(0{,}50(220(220)}$

$\alpha = 18°$ $\qquad\qquad$ $\alpha = 162°$

$\hat{B} = \hat{C} = 81°$ $\qquad\qquad$ $\hat{B} = \hat{C} = 9°$

Activity 1

LC SUMMATIVE ASSESSMENT

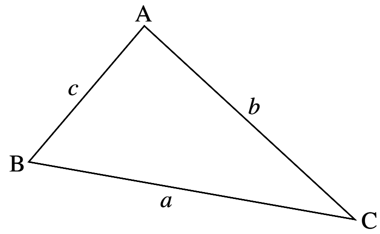
LC INDIVIDUAL

LC DVD

How do we find all the angles of a triangle that is not a right angled triangle?

We use the sine rule which tells us that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Let's prove it

Area $\triangle ABC = \frac{1}{2}$ bc sin A

Similary if B were at the origin

Area $\triangle ABC = \frac{1}{2}$ ac sin B

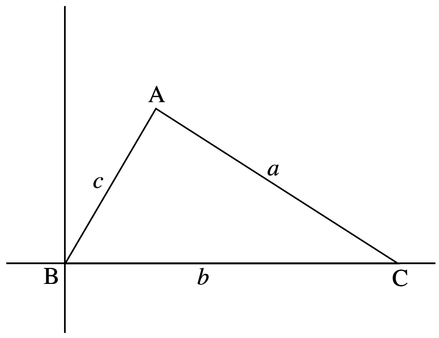
And if C were at the origin

Area $\triangle ABC = \frac{1}{2}$ ab sin C

But this is the same triangle

So $\quad \frac{1}{2}$ ab sin c $= \frac{1}{2}$ bc sin A $= \frac{1}{2}$ ac sin B

$\div \frac{1}{2}$ abc



$$\frac{\frac{1}{2}ab\sin C}{\frac{1}{2}abc} = \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}abc}$$

$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

Let's use it

Example 4:

Solve $\triangle ABC$

$\hat{B} = 59°$

$\frac{c}{\sin 100°} = \frac{12}{\sin 21°}$

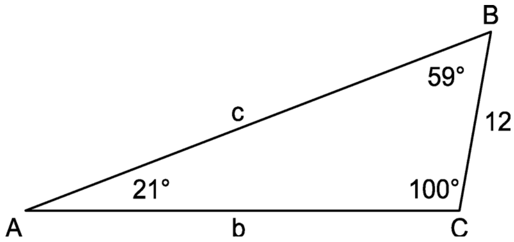
$c = \frac{12 \sin 100°}{\sin 21°}$

c = 33cm

$\frac{b}{\sin 59°} = \frac{12}{\sin 21°}$

$\therefore\ b = \frac{12 \sin 59°}{\sin 21°}$

$\therefore\ b = 28{,}7\text{cm}$



To apply the law of sines, you need to have information about two sides, and one angle opposite one of those sides, or two angles and the one side opposite one of the angles.

Activity 2

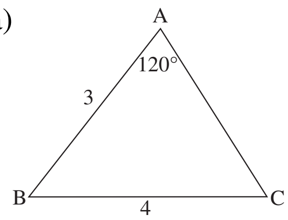
SUMMATIVE ASSESSMENT

INDIVIDUAL

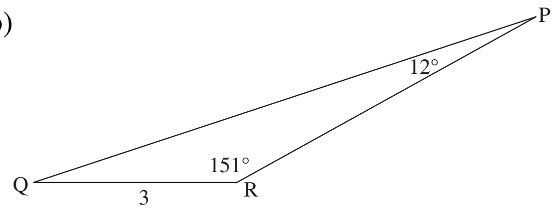
DVD

1. In the following two diagrams, calculate the area of each triangle, and also find all the other angles and sides using trigonometry:

a)



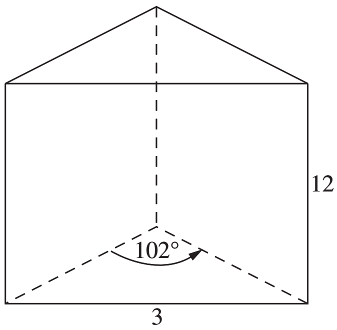
b)



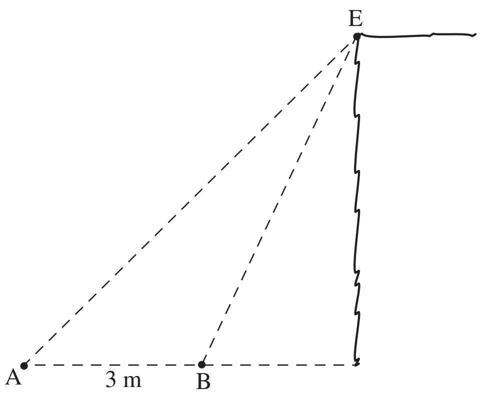
2. In the diagram alongside, an isosceles triangular based prism of height 12 cm is shown. The angle included by the isosceles sides is 102°.

a) Calculate the length of the hypotenuse of the base triangle.

b) Find the volume of the prism.



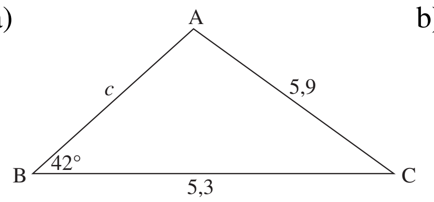
3. Two observers are 3 m apart from one another.



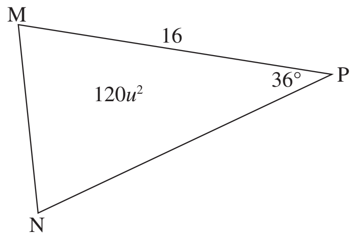
They are both deserving an eagle in its nest at the top of a rock face (at E). The angle of elevation of observer A is 75° and that of observer B is 81°. If the eagle would fly directly to observer B, how far would he have to fly?

4. Solve the triangle below:

a)

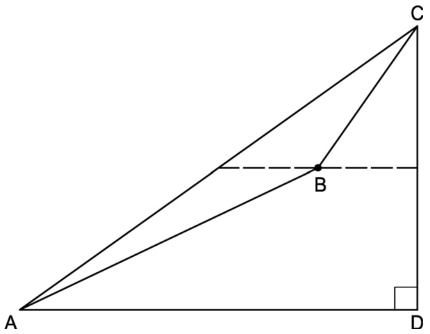


b)



Activity 3

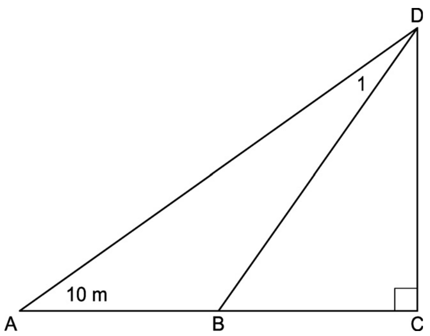
1. Look at this diagram.



At A Grant notices that the angle of elevation of the top cliff DC is 62°. He walks at an incline of 21° to point B which is 80m form A. He then notices that the angle of elevation of the top of the cliff is 78°

Find the height of the cliff.

2.



DC is a tower. The angle of elevation from A to the top of the tower is 43° and from B to the top of the tower is 74°. The distance AB is 10 m.

Find the height of the tower.

SOLVING TRIANGLES

Lesson 23

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape, space and measurement
Assessment Standard
Solve problems in two dimensions by using the sine, cosine and area rules, and by constructing and interpreting geometric and trigonometric models.

Overview

In this lesson you will:

- Discover how sometimes you need to draw two different triangles to solve the problem correctly.
- Learn to use and prove the cosine rule.
- Apply the sine, cosine and the area rule to real life problems.

More about the sine rule

Lesson

DVD

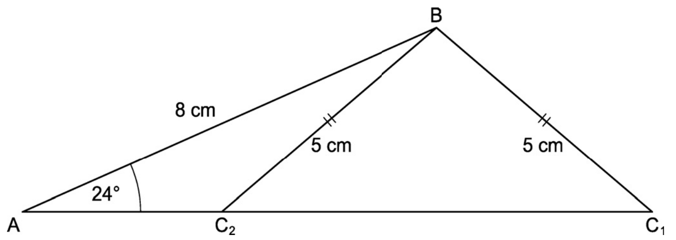
What happens if you have an angle and two sides, but **the side is opposite the given angle** is shorter than the other given side.

Let's investigate.

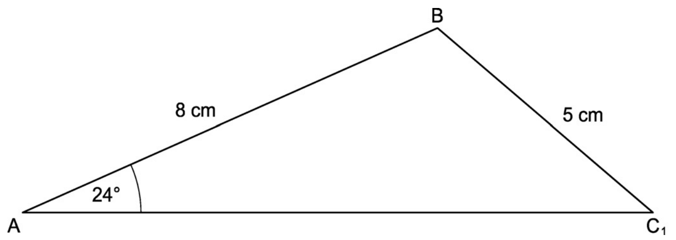
Solve $\triangle ABC$ if A = 24°

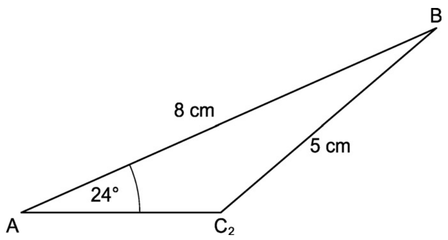
AB = $\hat{8}$cm and BC = $\hat{5}$cm

Look we have two possible triangles.



Let's draw them next to each other.





$$\frac{\sin C}{8} = \frac{\sin 24^\circ}{5}$$

$$\sin C = \frac{8\sin 24^\circ}{5}$$

C = 40,6° or C = 139,4°

$\therefore$ B = 115,4° or B = 16,6°

Now:

$$\frac{b}{\sin 115{,}4^\circ} = \frac{5}{\sin 24^\circ}$$

$$b = \frac{5\sin 115{,}4^\circ}{\sin 24^\circ}$$

b = 11,1cm

or

$$\frac{b}{\sin 16{,}6^\circ} = \frac{5}{\sin 24^\circ}$$

$$b = \frac{5\sin 16{,}6^\circ}{\sin 24^\circ}$$

$$b = \widehat{3{,}5}\text{cm}$$

This is called the ambiguous case

It happens when you are given one angle and two sides, but the shorter of the two sides is opposite the given angle.

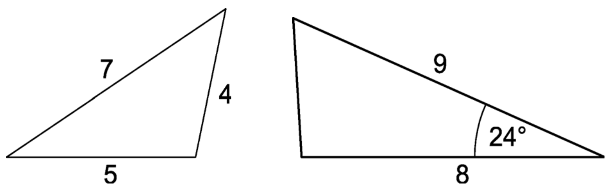
FORMATIVE ASSESSMENT

PAIRS

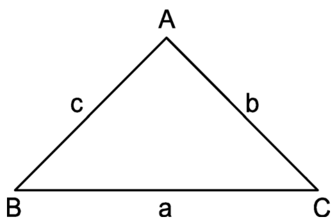
DVD

Activities 1–3

What happens if you do not have a side opposite the given angle?



We use the cos rule

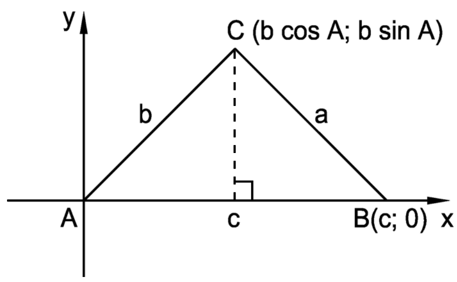


$b^2 = a^2 + c^2 - 2ac \cos \text{B}$

$a^2 = b^2 + c^2 - 2bc \cos \text{A}$

$\text{c}^2 = b^2 + a^2 - 2ab \cos \text{C}$

Let's prove it.



With A in standard position, the co-ordinates of apex C will be:

$x = b\cos \text{A}$

$y = b\sin \text{A}$

By the distance formula, we find the length of BC:-

$a^2 = (b \cos \text{A} - c)^2 + (\ \sin \text{A} - 0)^2$

$a^2 = b^2 \cos^2 \text{A} - 2bc \cos \text{A} + \text{c}^2 + \hat{b}^2 \sin^2 \text{A}$

$a^2 = b^2 \cos^2 \text{A} + b^2 \sin 2 \text{A} + \text{c}^2 - 2bc \cos \text{A}$

$a^2 = b^2(\cos^2 \text{A} + \sin^2 \text{A}) + \text{c}^2 - 2bc \cos \text{A}$ $\qquad \sin^2 \text{A} + \cos^2 \text{A} = 1$

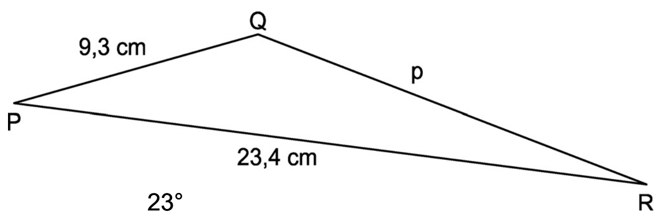
$a^2 = b^2 + c^2 - 2bc \cos \text{A}$

Examples

1. Solve $\triangle$PQR

$\text{p}^2 = (9{,}3)^2 + (23{,}4)^2 - 2(9{,}3)(23{,}4) \cos 23^\circ$

p = 15,3 cm



Now use the sine rule and solve for $\hat{R}$, because we have enough information to find this angle.

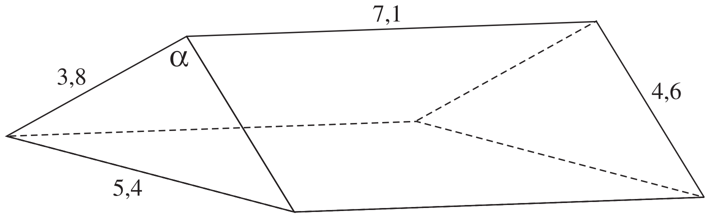
$\frac{\sin \hat{R}}{9,3} = \frac{\sin 23°}{15,3}$

$\sin \hat{R} = \frac{9,3 \sin 23°}{15,3}$

$\hat{R} = 13,7$

$\therefore\ \hat{Q} = 143,3°$

2. Find the volume of this triangular prism.



Volume = are of Base × height

To use the rule we need an angle.

The cos rule

$(5,4)^2 = (3,8)^2 + (4,6)^2 - 2(3,8)(4,6) \cos a$

$2(3,8)(4,6) \cos \alpha = (3,8)^2 + (4,6)^2 - (5,4)^2$

$\cos \alpha = \frac{(3,8)^2 + (4,6)^2 - (5,4)^2}{2(3,8)(4,6)}$

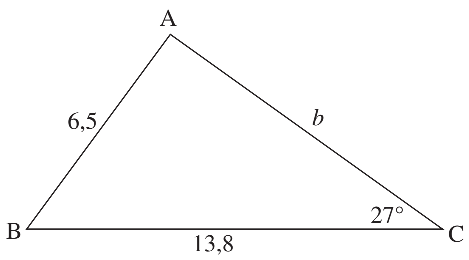
$\alpha = 79,4°$

Volume $= \frac{1}{2}(4,6)(3,8) \sin 79,4° \times \widehat{7,1} = 61\ \mu^3$

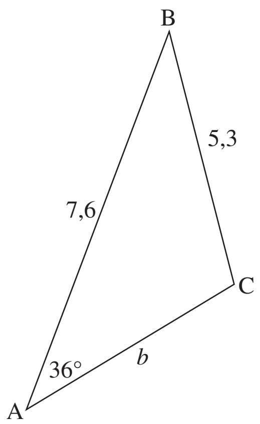
Activity 4

1. For each of the triangles below, find the possible lengths of side AC.

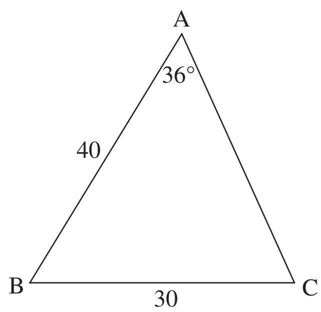
a)



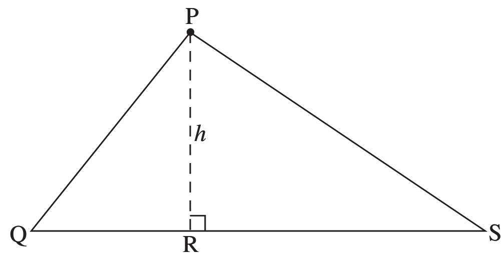
b)



c)

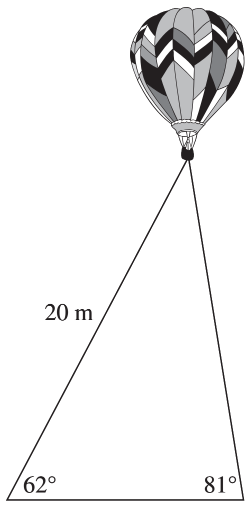


2. An observer is standing on a suspension bridge at point P. He wishes to measure the height of the bridge above ground level QRS. He measures the angles of depression to two beacans Q and S on the ground.



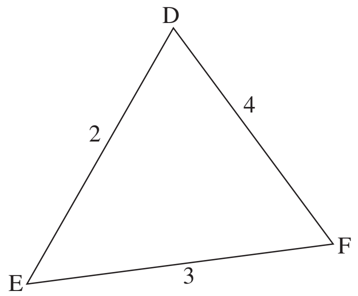
These beacans are 800 metres apart. To $\hat{Q}$ the angle of depression is 75° and to $\hat{S}$, it is 8°. How high is the suspension bridge above the ground.

3. A hot air balloon is kept at a constant altitude by two ropes anchoring it in the ground. One rope is 20 m long and makes an angle of 62° with the ground. If the second rope makes an angle of 81° with the gound, how long is this rope?

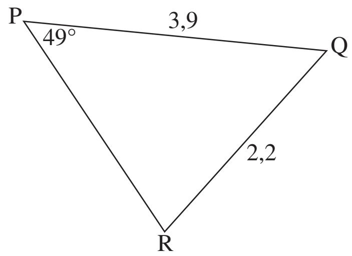


4. Solve the following triangles:

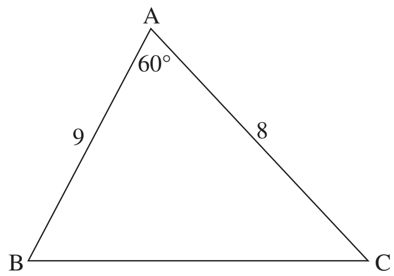
a)



b)



c)

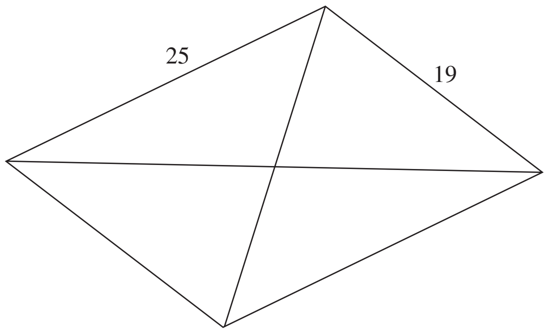


Activity 5

SUMMATIVE ASSESSMENT

INDIVIDUAL

1. The sides of a parallelogram are 25 cm and 19 cm and its longest diagonal is 33 cm long.

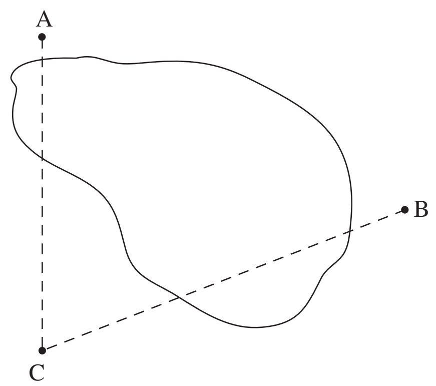


Find:

a) The angles of the parallelogram.

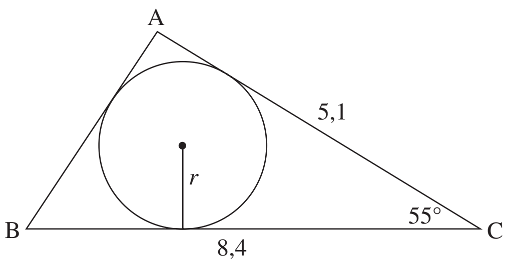
b) The length of the shortest diagonal.

2. Two houses are on the opposite side of a small lake, at A and B.



From point C, an observer notes that the angle between the lines of sight AC and BC is 35°. The distance from observation point to house A is 357 m and to house B is 291 m. What is the distance between the houses?

3. In ΔABC, $BC = 8{,}4$, $AC = 5{,}1$ and $\hat{B} = 55°$.



Find the a) area of ΔABC

b) The radius of the circle

(Problem taken from: www.acts.tinet.ie/area of triangle_673.html)

Lesson 24

SOLUTIONS OF TRIANGLES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape, space and measurement
Assessment Standard
Solve problems in two dimensions by using the sine, cosine and area rules, and by constructing and interpreting geometric and trigonometric models.

Overview

In this lesson you will:

- Look at geometric and trigonometric models in two dimensions and solve real life problems.
- Be required to use your calculator correctly.
- Need to apply the sine, cosine and the area rule

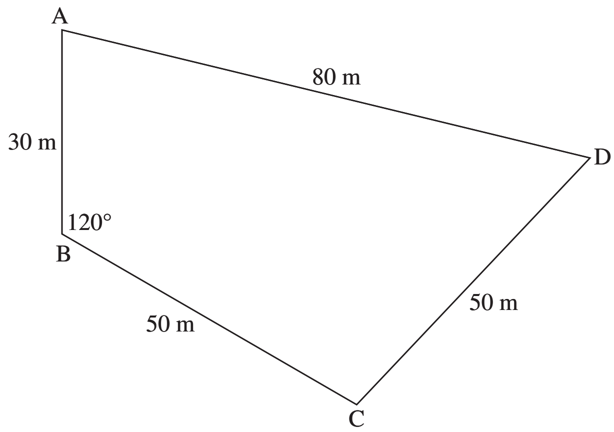
Lesson

DVD

Example 1

Look at the diagram.

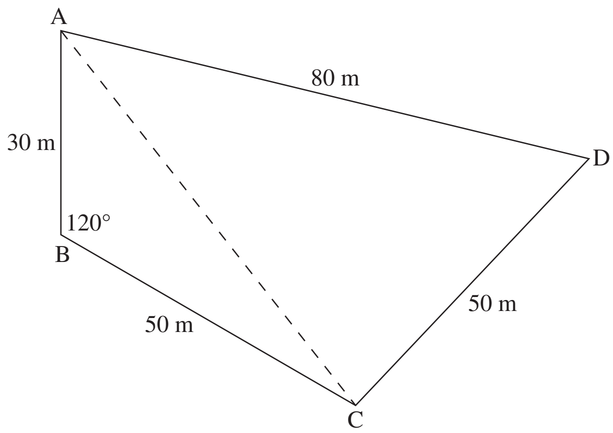
A, B, C and D are 4 beacons on a farm. Find the area of the land enclosed by these four beacons if $\hat{B} = 120°$.



Solution

To find the area of the quadrilateral ABCD, it will be wise to divide it into two triangles. That way we can use our method to determine the area of a non-right angled triangle.

So we join A and C. (If we join B and D, we will split up the angle B, and we do not know how this split happens). So now:



As we have two sides and the included angle, we can find the area of ΔABC.

So: Area $\Delta ABC = \frac{1}{2}$ AB.BC sin B

$= 649{,}5 \text{ m}^2$.

We do not have enough information in ΔADC, so we need to find AC first and use the cosine rule for this (since we have 3 sides), and find $\hat{D}$. After that, we can find the area.

So in ΔABC: $AC^2 = AB^2 + BC^2 - 2AB.BC.\cos \hat{B}$

$\therefore AC^2 = 30^2 + 50^2 - 2(30)(50) \cos 120°$

$= 4\,900$

$\therefore AC = 70$

In ΔADC: $AC^2 = AD^2 + DC^2 - 2AD.DC.\cos \hat{D}$

$4\,900 = 6\,400 + 2\,500 - 2(80)(50)\cos \hat{D}$

$\therefore \cos \hat{D} = \frac{1}{2}$

$\therefore \hat{D} = 60°$

$\therefore$ Area $\Delta ADC = \frac{1}{2} AD.DC.\sin \hat{D}$

$= \frac{1}{2}\, 80.50.\sin 60°$

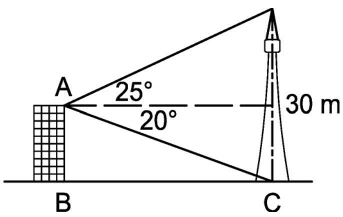
$= 1\,732 \text{ m}^2$

$\therefore$ Area of quadrilateral ABCD $= 649{,}5 + 1\,732 = 2\,381{,}6 \text{ m}^2$

Example 2

Look at this diagram

From a point A on top of a building, the angle of elevation to the top of a tower is 25° and the angle of depression to the foot of the tower is 20°. If the height of the tower is 30m, how far is the building from the tower, if they lie in the same horizontal plane.



Solution

Let's put in the information.

We need to find BC.

First we must find AC.

In ΔADC

$\frac{AC}{\sin 65°} = \frac{30}{\sin 45°}$

$AC = \frac{30 \sin 65°}{\sin 45°}$

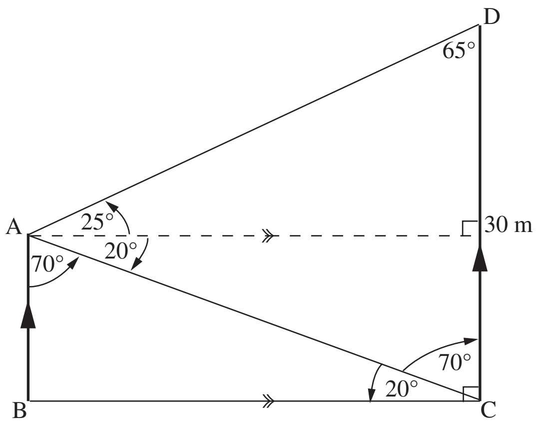
$AC = 38{,}5$m

Now in $\triangle ABC$

$\hat{B} = 90°$:

So $\cos 20° = \frac{BC}{AC}$

$\therefore BC = \cos 20 \times 38{,}5 \text{ m} = 36{,}1 \text{ m}$



Activities 1–5

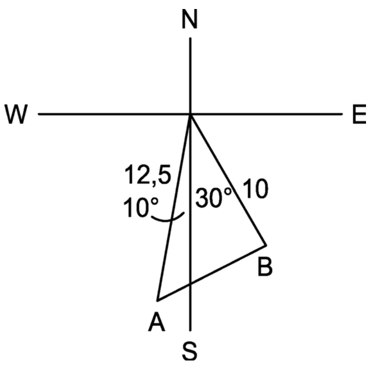
Example 3

Now you need to draw your own diagram

Two pleasure cruisers leave the same pier simultaneously, one travelling at 20 km/h in a direction S30°E and the other at 25 km/h in a direction S10°W.

How far apart will they be after 30 minutes?

Solution



Start south and move 30° East.

Then start south and move

West 10°

Distance = speed × time

1st Cruiser = $20 \times \frac{1}{2}$ = 10 km

2nd Cruiser = $25 \times \frac{1}{2}$ = 12,5 km

We need to find AB

$AB^2 = (10)^2 + (12{,}5)^2 - 2(10)(12{,}5)\cos 40°$

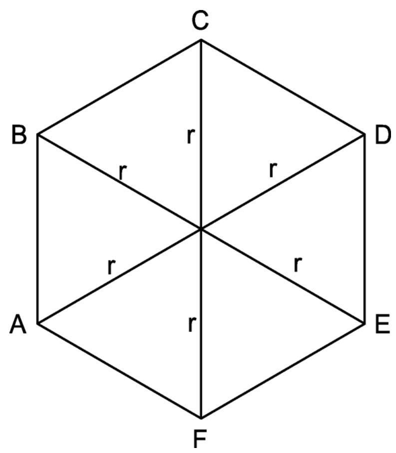
AB = 8 km

Example 4 (Formula's and problem solving.)

Look at this regular hexagon.

r is the radius of the circumscribed circle.

Find a formula for the area of the hexagon.



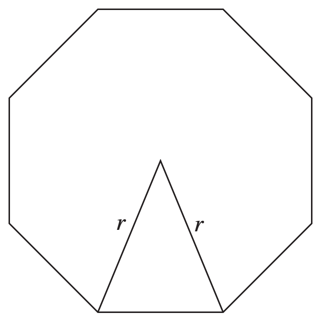
Solution

The central angle is $\frac{360°}{6} = 60°$

$\therefore$ area $= 6 \times \frac{1}{2} r \cdot r \sin 60°$

$= \frac{3r^2 \cdot \sqrt{3}}{2}$

Find the area of a regular octagon with radius of the circumscribed circle being r.



centre angle: $\frac{360°}{8} = 45°$

There are 8 iscoceles triangles in this regular octagon.

So area $= 8 \times \frac{1}{2} r.r.\ \sin 45°$

$= 4r^2 \frac{1}{\sqrt{2}}$

$= \frac{4r^2}{\sqrt{2}}$

Hence, find a formula for the area of a regular polygon with *n* sides

Solution

The central angle is $\frac{360°}{n}$

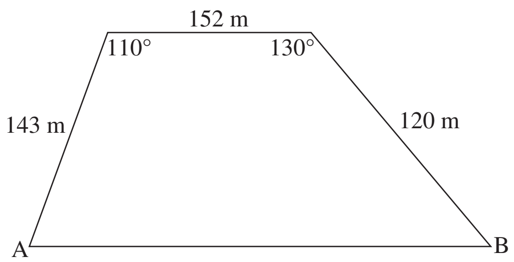
Area $= n \times \frac{1}{2} r^2 \cdot \sin \frac{360°}{n}$

Activity 1

SUMMATIVE ASSESSMENT

INDIVIDUAL

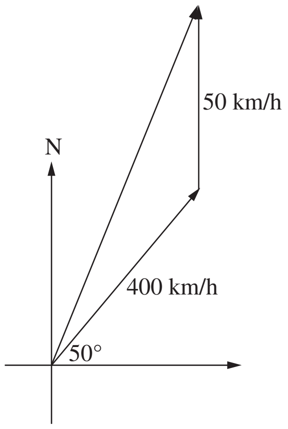
1.



a) Find $\hat{A}$.

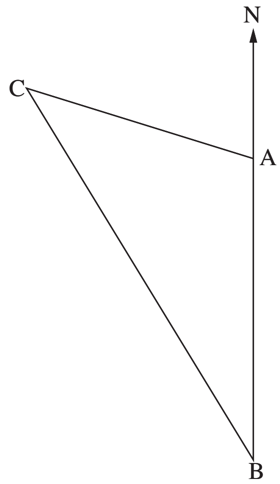
b) Determine the length of AB

2.



A plane travelling at 400 km/h is flying with a bearing of 40°. There is a wind of 50 km/h from the South. If no correction is made for the wind, what are the final bearing and ground speed of the plane?

3. A reconnaissance plane leaves an aircraft carrier at A and flies due south at a steady speed of 500 km/h. The carrier meanwhile proceeds at 30 km/h on a coarse N 60° W (or a bearing of 300°). The plane has fuel for only four hours flying and the intention is to use all the fuel. This means that after flying to a certain point B, it must turn and fly in the direction BC to intercept the carrier at C before its fuel is exhausted.



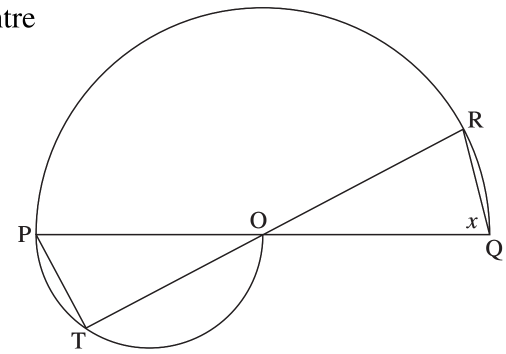
3.1 What is the distance of AC

3.2 What total distance does the plane fly

3.3 Use the rule of cosines to write down an equation in x and hence find, to the nearest kilometre, the maximum distance South that the plane can fly before turning to rejoin the ship.

3.4 Calculate the angle B and hence give the direction BC in which the plane must fly in order to meet the carrier at C.

4. In the figure alongside, O is the centre of the semi-circle PRQ with radius r. PO is a diameter of the semicircle PTO. Angle Q = x.



Determine:

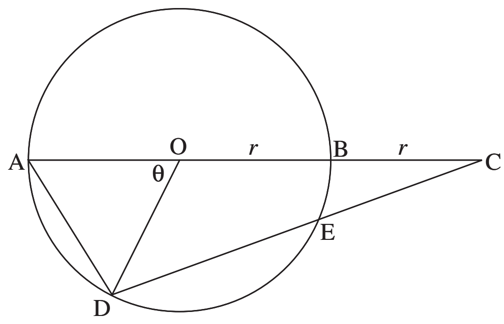
4.1 RQ in terms of r and x by using the sine rule, and simplify the expression if $\sin 2x = 2 \sin x \cos x$

4.2 The area of ΔROQ in terms of r and x

4.3 The area of ΔPOT in terms of r and x

4.4 The ratio of $\frac{\text{Area } \Delta ROQ}{\text{Area } \Delta POT}$ if $x = 75°$, leaving your answer in surd form.

5. In the figure, AB is a diameter of circle O with radius equal to r. AB is produced to C such that BC = r. CED is a secant such that CE = ED = DA and angle AOD = θ.



5.1 Prove that: $\cos \theta = \frac{1}{4}$

5.2 Hence prove that the area of $\Delta AOD = \frac{r^2\sqrt{15}}{8}$

6. A tower PQ stands on top of a hill QR. At a point A in the same horizontal as R, the angle of elevation of P, the top of the tower is 2θ and the angle of the elevation of Q, the base of the tower, is θ. The height of the tower is a.

6.1 Show that $AP = \frac{a}{\tan \theta}$.

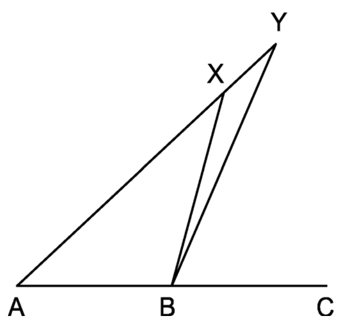
6.2 Hence show that the height of P above the horizontal plane, is $2a \cos^2 \theta$ if $\sin 2\theta = 2 \sin \theta = 2 \sin \theta \cos \theta$.

6.3 If $\theta = 30°$, calculate PQ/PR.

7. A shadow, cast on a lake by a stationary balloon, moves by 12 m as the angle of elevation of the sun increases form 62° to 70°. Determine the height of the balloon.

8. A man walking down a straight road ABC notice two trees X and Y. he also notice that A, X and Y lie in a straight line and $X\hat{A}C = 43°$. He walks 800 m to B, and observes that $X\hat{B}C = 74°$ and $Y\hat{B}C = 66°$.

Calculate the distance between the two trees.



AREAS AND VOLUMES

Lesson 25

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape, space and measurement
Assessment Standard
- Surface area and volume of right pyramids and cones.
- Volumes of spheres and hemispheres.

Overview

In this lesson you will:

- Review what you learnt in Grade 10.
- Discover the volume of a right pyramid.
- Discover the surface area of a right pyramid.
- Apply your knowledge to problem solving.

Lesson

LC DVD

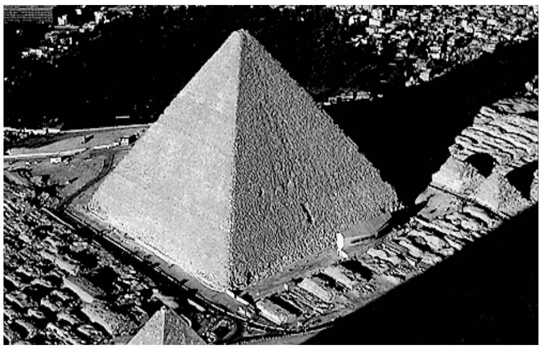
LC BASELINE ASSESSMENT

LC PAIRS

Activity 1

Grade 11 work

Pyramids



The Great Pyramid of Giza

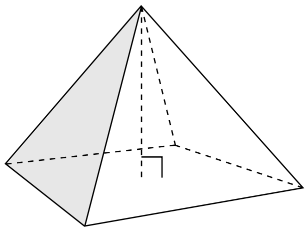
It is the one and only Wonder which does not require a description by early historians and poets. It is the one and only Wonder that does not need speculations concerning its appearance, size, and shape. It is the oldest, yet it is the only surviving of the Seven Ancient Wonders.

Defining the Pyramid

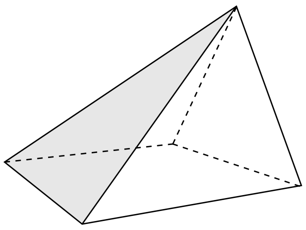
A pyramid is a polyhedron with one face (known as the "base") a polygon and all the other faces triangles meeting at a common polygon vertex (known as the "apex").

(A polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges)

A right pyramid is a pyramid for which the line joining the centroid of the base and the apex is perpendicular to the base.

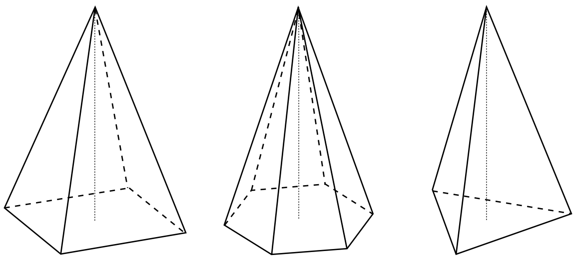


right pyramid



non-right pyramid

A regular pyramid is a right pyramid whose base is a regular polygon.



Formulae that we will use:

Volume of any regular pyramid = $\frac{1}{3}$ × base area × height

a) The triangular based pyramid:

1. The pyramid has 4 faces
2. There are three side faces which are triangles
3. The base is a triangle
4. The pyramid has four vertices
5. The pyramid has six edges

Surface Area = [Base Area] + $\frac{1}{2}$ × Perimeter × [Slant Height]

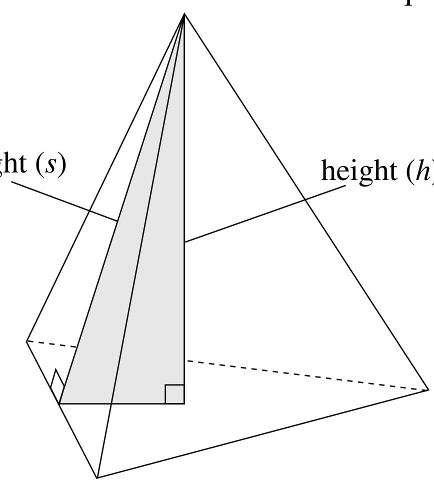
Volume = $\frac{1}{3}$ × [Base Area] × Height

- **If all the edges are equal, we have a tetrahedron for which:**

  Surface Area = $\sqrt{3} \times (\text{side length})^2$

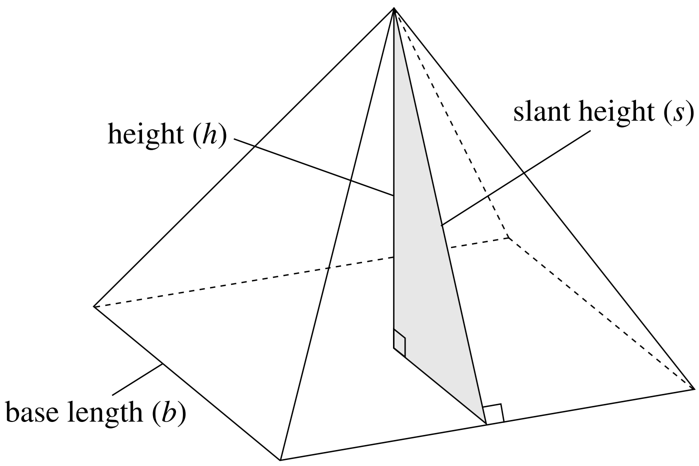
  Volume = $\left(\frac{\text{side length}}{6\sqrt{2}}\right)^3$

- All four of the vertices in a tetrahedron are equidistant from one another



LC Maths G11 WB.indb 108 2008/08/09 08:25:17 AM

b) A square based pyramid:



1. The pyramid has 5 faces
2. There are four side faces which are triangles
3. The base is a square
4. The pyramid has five vertices
5. The pyramid has eight edges

Surface Area = [Base Area] + $\frac{1}{2}$ × Perimeter × [Slant Height]

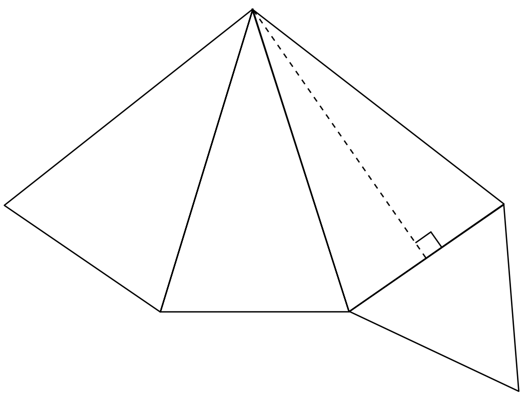
Volume = $\frac{1}{3}$ × [Base Area] × Height

An interesting observation:

Number of base sides	Base polygon	Number of faces	Number of side faces	Number of vertices	Number of edges
3	Triangle	4	3	4	6
4	Square	5	4	5	8
5	Pentagon	6	5	6	10
6	Hexagon	7	6	7	12
7	Heptagon	8	7	8	14
8	Octagon	9	8	9	16

Where do the surface area formulas come from?

Let us look at the nets for the pyramids:



For the triangular based pyramid we have three sides each with an area of $\frac{1}{2}$ × base × height = $\frac{1}{2} \times b \times s$.

So for the three sides we will have $3 \times \frac{1}{2} \times b \times s = \frac{3bs}{2}$. Then for the base we have one of two options:

Area rule: $\frac{1}{2} \times b \times \sin 60° = \frac{b^2}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}b^2}{4}$

Or height is $\sqrt{b^2 - \left(\frac{b}{2}\right)^2} = \sqrt{\frac{3}{4}b^2} = \frac{\sqrt{3}b}{2}$ and thus the area of the base $= \frac{1}{2} \times \frac{\sqrt{3}b}{2} \times b$ $= \frac{\sqrt{3}b^2}{4}$.

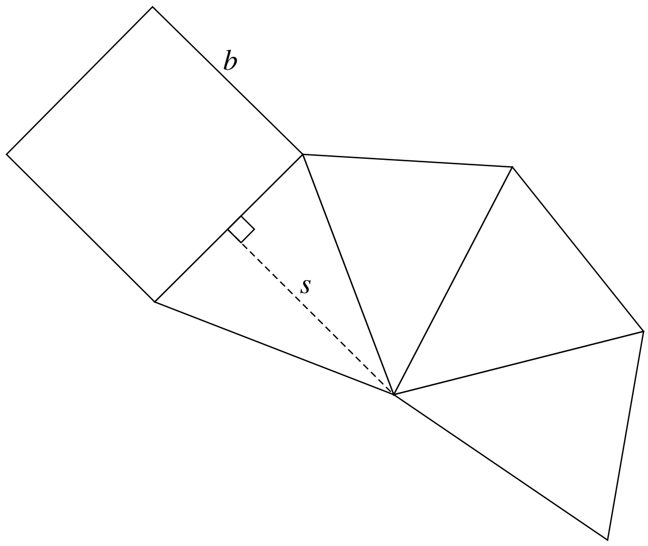
Thus the total surface area of the triangular based pyramid will be $\left(3 \times \frac{1}{2} \times b \times s\right) + \left(\frac{1}{2} \times \frac{\sqrt{3}\text{b}}{2} \times \text{b}\right) = \frac{3\text{sb}}{2} + \frac{\sqrt{3}\text{b}^2}{4}$.

That is equivalent to: Surface area of pyramid = Area of base + $\frac{1}{2}$(perimeter of base × slant height).

Activity 1

Draw a net of the square based pyramid with base length b and slant height s and use the net to show that the formula Surface area of pyramid = Area of base + $\frac{1}{2}$ (perimeter of base × slant height) holds true in this case.

Solution Activity 1:



There are four sides which are triangular with base length b and height s. So the area of the four will collectively be $4 \times \frac{1}{2} \times$ base × height $= 2bs$. The area of the base is b^2. So the total surface area of the square based pyramid will then be given by $2bs + b^2$.

Let's compare this with Area of base + $\frac{1}{2}$(perimeter of base × slant height):

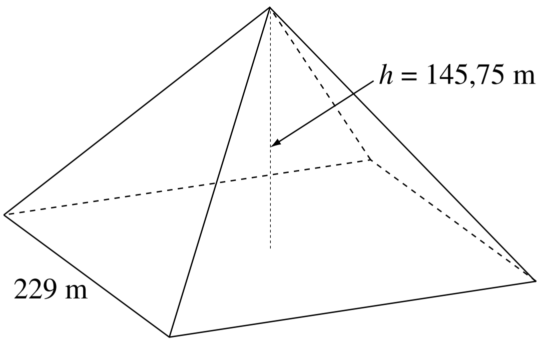
Surface area of pyramid = Area of base + $\frac{1}{2}$(perimeter of base × slant height)

$$= b^2 + \frac{1}{2} \times 4bs \times s$$

$$= b^2 + 2bs.$$

So the formula holds true.

Example 1:



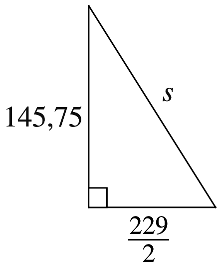
In Egypt, the Great Pyramid of Giza is 145,75 m in height and has a square base of 229 m on a side. The triangular sides angle of 51,85° with the square base.

Find the

a) Total surface area and b) volume of the great pyramid

Solution:

a) Slant height is what we need to find first:



$s = \sqrt{145{,}75^2 + \left(\frac{229}{2}\right)^2} = 185{,}35$

Or alternatively:

$s = \frac{145{,}75}{\sin 51{,}85^\circ} = 185{,}34$

Surface area of pyramid = Area of base + $\frac{1}{2}$(perimeter of base × slant height)

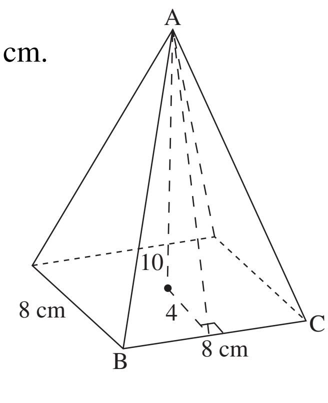
$= (229)^2 + \frac{1}{2}(4 \times 229 \times 185{,}35)$

$= 137\ 331{,}3\ \text{m}^2$

b) Volume of the pyramid = $\frac{1}{3} \times$ (Base Area) × Height = $\frac{1}{3} \times 229^2 \times 145{,}75 =$ 2 547 758,58 m^3

Example 2

Find the surface area and volume of this pyramid with a height of 10 cm and a square base with side 8 cm.



Solution

Volume = $\frac{1}{3}$ area of base × height

$= \frac{1}{3} \times (64)(10)$

$= 213{,}3$

Surface area = area of base + 4 triangles

Height of △ABC is found by Pythagoras:

$h^2 = 4^2 + 10^2$

$\therefore h^2 = 116$

$\therefore h = 10{,}8$

Area △ABC = $\left(\frac{1}{2}\right)(8)(10{,}8) = 43{,}1$

$\therefore$ Surface Area = (8)(8) + 4(43,1)

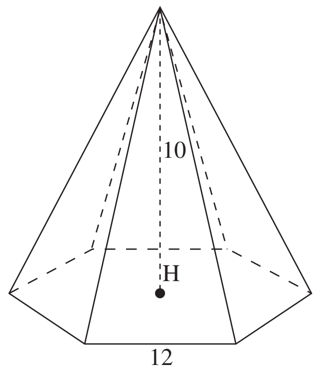
= 236,3 cm^2

Example 3

Find the

a) volume and

b) surface area if the side of the base hexagon is 12cm, and the height (H) of the pyramid is 10 cm.



Solution

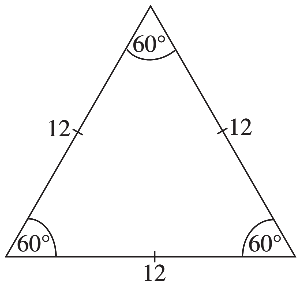
a) $V = \frac{1}{3}$ area of base × 10

The base divides into 6 congruent △s

so area of base

$= 6 \times \left(\frac{1}{2}\right)(12)(12) \sin 60°$

$= 374{,}1 \text{ cm}^2$



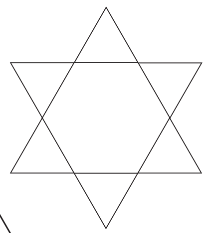
Thus:

$\text{Volume} = \frac{1}{3}.374{,}1 \times 10$

$= 1\ 247 \text{ cm}^3$

b) Surface area

= Area of 6 congruent △s + area of hexagonal base



Base:

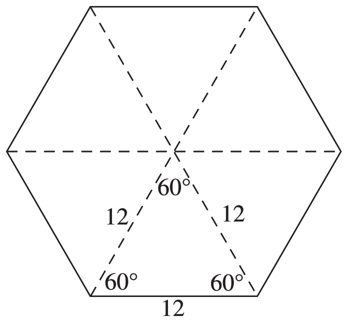
$\text{Area} = 6 \times \frac{1}{2}(12)(12) \sin 60°$

$= 3(144) \sin 60°$

$= 374{,}12 \text{ cm}^2$

$p^2 = 36 + 144 = 180$

$p = 13{,}42$ cm

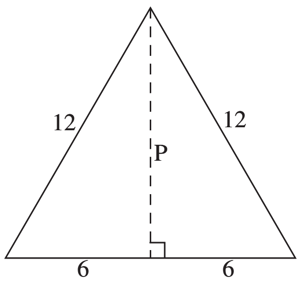


For the triangles:

The height of each Δ in the base.

$p^2 = 144 - 36 = 108$

$\therefore p = 10{,}39$



The slant height which will give the height of each Δ in the sides.

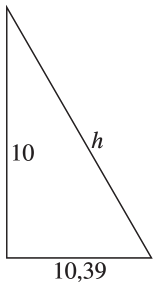
Now:

$h^2 = 10^2 + (10{,}39)^2$

$= 208$

$\therefore h = 14{,}4$

Each Δ in the sides.



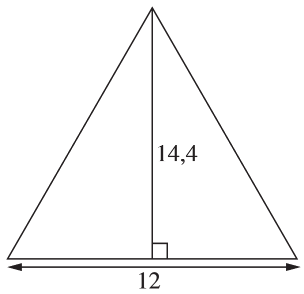
Now finally:

$6 \times \frac{1}{2}(12)(14{,}4)$

$= 518{,}4$

$\therefore$ Total surface area $= 374{,}12 + 518{,}4$

$= 892{,}52 \text{ cm}^2$



Activity 2 No 1, 2

FORMATIVE ASSESSMENT

PAIRS

DVD

Example 4

If it is given that the slant height of a pyramid is 22 cm and the base length is 14 cm, find the surface area and the volume of each pyramid if

a) The base is an equilateral triangle

b) The base is a pentagon

Solutions:

a) Surface area of pyramid

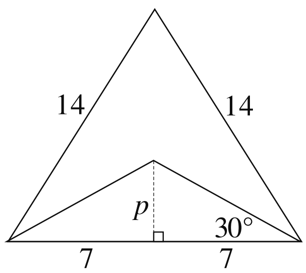
$= \text{Area of base} + \frac{1}{2}(\text{perimeter of base} \times \text{slant height})$

$= \frac{1}{2} \times 14^2 \times \sin 60° + \frac{1}{2}(3 \times 14 \times 22)$

$= 49\sqrt{3} + 462$

$= 546{,}87 \text{ cm}^2$

For the volume we need the height from the base to the apex of the pyramid.



From the top view of the base we get:

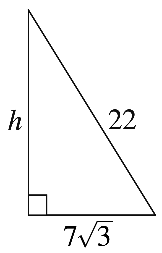
$p = \frac{7}{\tan 30} = 7\sqrt{3}$

The height is then:

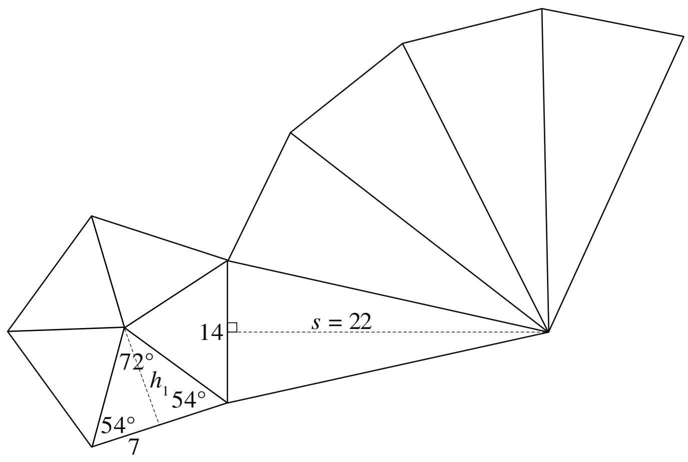
$h = \sqrt{22^2 - (7\sqrt{3})^2} = 18{,}36$

Volume of the pyramid $= \frac{1}{3} \times (\text{Base Area}) \times \text{Height}$

$= \frac{1}{3} \times \left(\frac{1}{2} \times 14^2 \times \sin 60°\right) \times 18{,}36 = 519{,}41 \text{ m}^3$



b)



For the base:

$$h = 7\tan 54° = 9{,}635 \text{ cm}$$

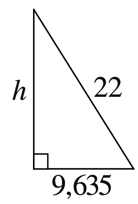
$$\text{Thus base area} = 5 \times \frac{1}{2} \times 14 \times 9{,}365$$

$$= 337{,}23 \text{ cm}^2$$

$$\text{Area sides} = 5 \times \frac{1}{2} \times 14 \times 22 = 770 \text{ cm}^2$$

$$\text{Total surface area} = 770 + 337{,}23$$

$$= 1\ 107{,}23 \text{ cm}^2.$$



For the height of the pyramid:

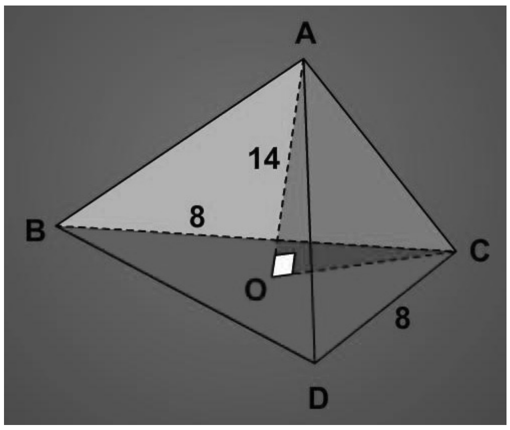
$$h = \sqrt{22^2 - 9{,}635^2} = 19{,}78$$

$$\text{Volume of the pyramid} = \frac{1}{3} \times (\text{Base Area}) \times \text{Height}$$

$$= \frac{1}{3} \times 337{,}23 \times 19{,}78 = 2\ 223{,}24 \text{ cm}^3.$$

Example 5

Find the volume and surface area of this pyramid with a triangular base of 8 cm and a height of 14 cm.



Solution

$$\text{Volume} = \frac{1}{3} \text{ base area} \times \text{h}$$

$$\text{area of base} = \frac{1}{2}(8)(8) \sin 60°$$

$$= 27{,}713 \text{ cm}^2$$

$$\therefore \text{Vol} = \frac{1}{3}(27{,}713)(14)$$

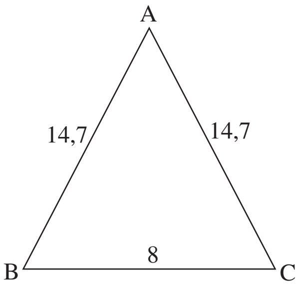
$$= 129{,}3 \text{ cm}^3$$

Surface area = area of base + area 3 triangles

We need OC (on the ground)

Then we use Pythagoras to get AC.

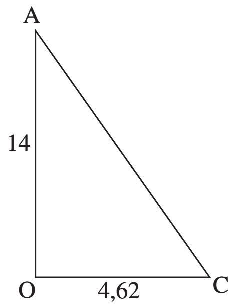
OC = 4,6



In ΔOPC:

$\cos 30° = \frac{PC}{OC}$

$\therefore OC = \frac{4}{\cos 30°} = 4{,}62$ cm



Now in ΔAOC:

$AC^2 = 14^2 + (4{,}62)^2$

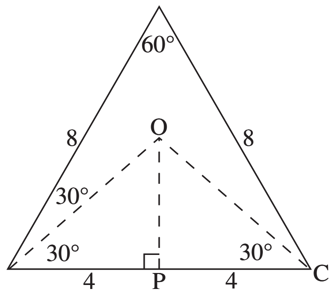
$\therefore AC = 14{,}7$

By the cos rule:

$8^2 = (14{,}2)^2 + (14{,}7)^2 - 2(14{,}7)(14{,}7) \cos A$

$2(14{,}7)(14{,}70 \cos A = (14{,}7)^2 + (14{,}7)^2 - 8(2)$

$\cos A = \frac{14{,}7^2 + 14{,}7^2 - 8^2}{2(14{,}7)(14{,}7)}$



$\hat{A} = 31{,}6°$

Surface area = 55,4 + 391/2)(14,7)(14,7) sin 31,6 = 225,2 cm^2

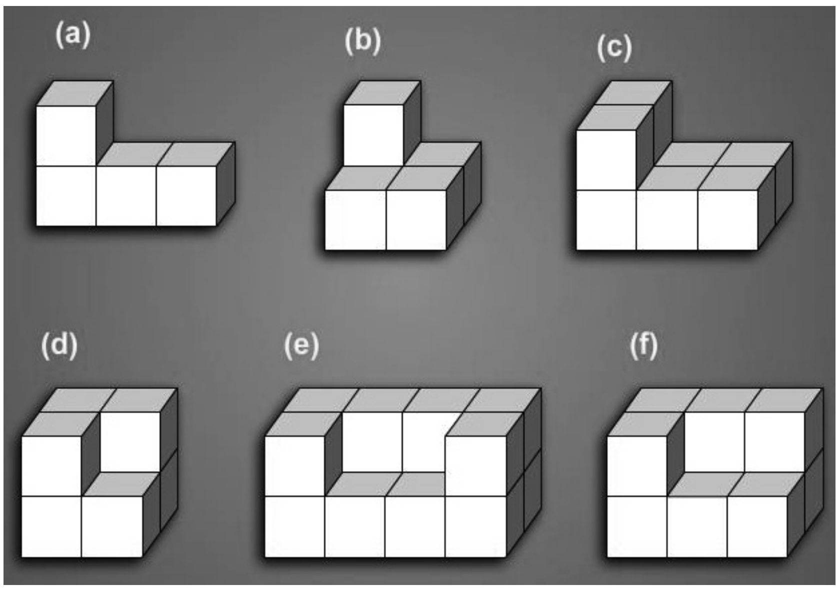
Activity 2 No 3 – 6

SUMMATIVE ASSESSMENT

Activity 1

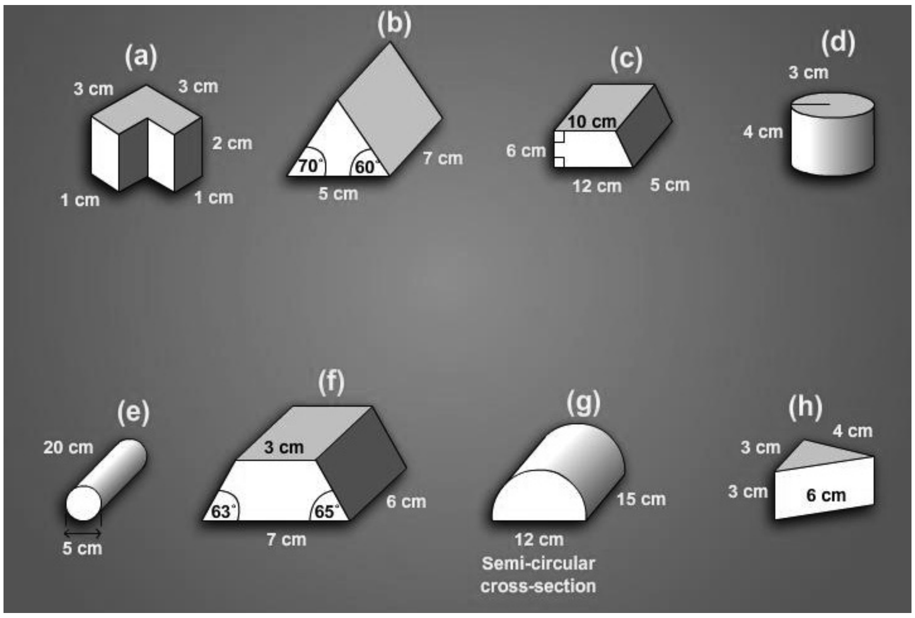
INDIVIDUAL

1. Shapes are made using one-centimetre cubes. Find the volume and surface are of each shape.



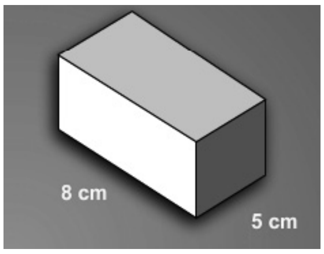
2. Find the volumes of these prisms.

Where necessary take π to be 3, 14 or use the π key on your calculator.



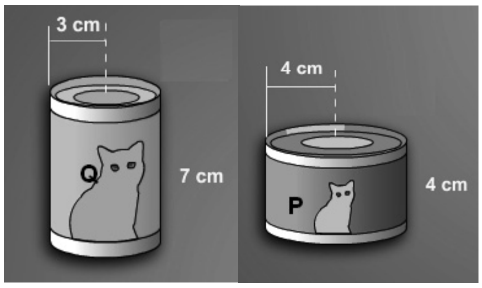
3. (a) The surface area of this cuboid is 197 cm^2

Calculate the volume of the cuboid.



(b) Which tin holds more cat food?

(c) A cylinder with a radius of 3 cm and a height of 8 cm is full of water. The water is poured into another cylinder with a diameter of 8 cm. Calculate the height of the water.

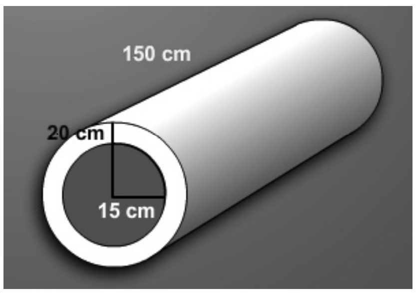


(d) A cylinder has a radius of 3,6 cm. the volume of the cylinder is 3346 cm^3. Calculate the total surface area of the cylinder. Give your answer to an appropriate degree of currency

(e) A concrete pipe is 150 cm long. It has an internal radius of 15 cm and an external radius of 20 cm. Calculate, giving your answer to 3 significant figures,

a) the area of the curved surface inside of the pipe.

b) the curved surface area of the outside of the pipe.



Activity 2

FORMATIVE ASSESSMENT

PAIRS

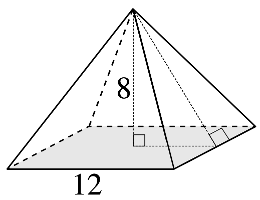
1. A pyramid of Khufu is a regular square pyramid with a base edge of approximately 776 feet and an original height of 481 feet. The limestone used to construct the pyramid weighs approximately 167 pounds per cubic foot.

   a) Estimate the weight of the pyramid of Khufu in pounds. (Assume the pyramid is a solid.)

   b) If 1 pound = 0.453 592 37 kilogram, what is the weight of the pyramid in kg?

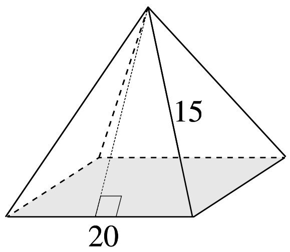
2. a. Find the length of the slant height given the base edge is 12cm and the height of the pyramid is 8cm.

   Also find the Surface area and the Volume of the pyramid.



   b. Find the length of the slant height given that the base edge is 20 cm and the lateral edge is 15 cm.

   Also find the Surface area and the Volume of the pyramid.



3. Suppose that the height of a regular square pyramid is 3 cm and the length of one edge is 5 cm. What are the surface area and volume of this pyramid?

4. Investigate what will happen to the area and volume of any square based pyramid where the base length is *b* and the height is *h*,

   i) if the base length is

      a) Halved

      b) Doubled

   ii) if the height is

      c) Halved

      d) Doubled

AREAS AND VOLUMES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
Surface area and volume of right pyramids and cones. Volumes of spheres and hemispheres.

Overview

In this lesson you will:

- Use the volume of a cone to solve problems.
- Find the curved surface area of a cone.
- Use the surface area and volume of a sphere to solve problems.
- Apply this knowledge to real life problems.

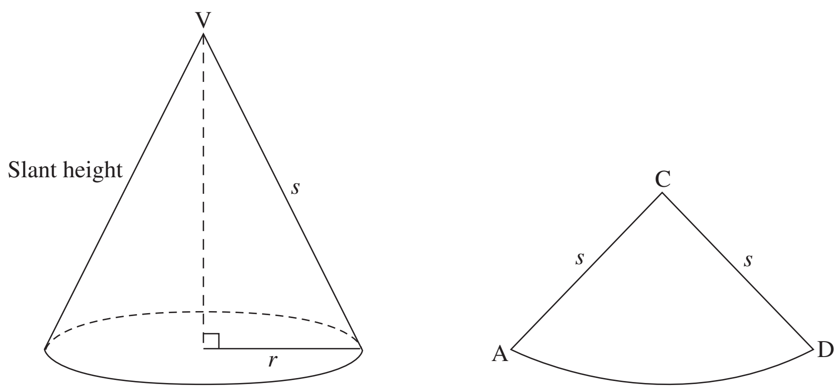
Lesson

DVD

Total surface area of a cone

If a cone of base radius r and slant height s is cut along the slant height and opened out flat, then the radius of the sector formed is s and the arc length AB is $2\pi r$.

The Sector ABC:



Circumference of the base $= 2\pi r$

Length of arc AB $= 2\pi r$

Now, $\dfrac{\text{Area of sector ABC}}{\text{Area of circle with centre at C}} = \dfrac{\text{Arc length AB of sector ABC}}{\text{Circumference of circle with centre at C}}$

$$\therefore \frac{\text{Area of sector ABC}}{\pi s^2} = \frac{2\pi r}{2\pi s} = \frac{r}{s}$$

Area of sector ABC $= \frac{r}{s} \times \pi s^2$

$= \pi rs$

If the curved surface area of a cone is equal to the area of sector ABC, then:

The area of the curved surface of a core $= \pi rs$

$\therefore$ Total surface area of the cone = Area of the base + Area of curved surface

$\therefore$ SA $= \pi r^2 + \pi rs$

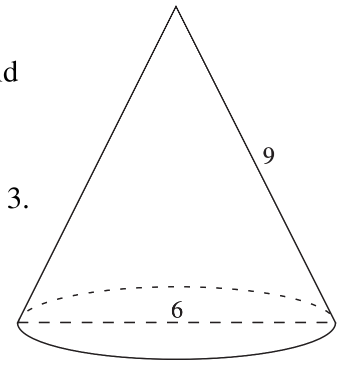
$= \pi r(r + s)$

Volume of a cone $= \frac{1}{3}$ base area $\times h$

$= \frac{1}{3}\pi r^2 h$

Examples

1. Calculate the surface area and volume of this solid



Solution

If the base diameter is 6, then the base radius (r) will be 3.

So: $s = 9$ and $r = 3$

Total surface area

$= \pi r(r + s)$

$= \pi(3)(3 + 9)$

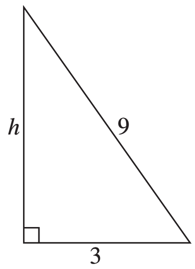
$= 36\pi$ cm^2

$= 113{,}1$ cm^2

For the volume:– $\frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(3)^2 h$

Now AW $h^2 = 9^2 - 3^2$



$\therefore\ h = 8{,}5$

$\therefore$ Volume $= \frac{1}{3}\pi.9.8.5$

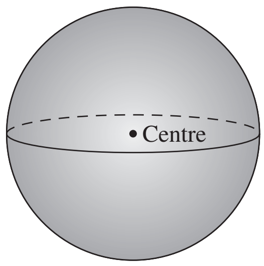
$= 79{,}97$

≈ 80 cm^3

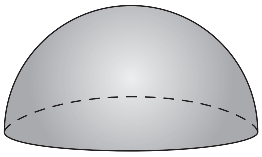
The sphere and the hemisphere

Surface area of a sphere

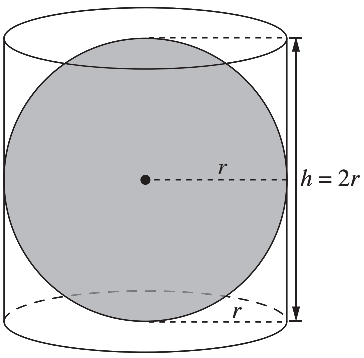
A sphere is a body bounded by a surface whose every point is **equidistant** (i.e. the same distance) from a fixed point, called the centre. For example, a shot (a heavy iron ball) is a solid sphere and a tennis ball is a hollow sphere.



One-half of a sphere is called a **hemisphere**.



Archimedes discovered that a cylinder that circumscribes a sphere, as shown in the following diagram, has a curved surface area equal to the surface area, S, of the sphere.



Surface area of sphere = curved surface area of a cylinder

$\therefore$ S $= 2\pi rh$

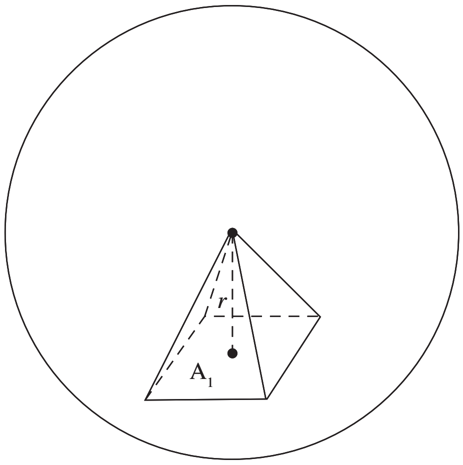
$= 2\pi r(2r)$

S $= 4\pi r^2$

Volume of a sphere

If four points on the surface of a sphere are joined to the centre of the sphere, then a pyramid of perpendicular height r is formed, as shown in the diagram. Consider the solid sphere to be built with a large number of such solid pyramids that have a very small base which represents a small portion of the surface area of a sphere.

If A_1, A_2, A_3, A_4, … A_n represent the base areas (of pyramids) on the surface of a sphere, then:



V = volume of the sphere

= Sum of the volumes of all pyramids

$= \frac{1}{3}A_1 r + \frac{1}{3}A_2 r + \frac{1}{3}A_3 r + \frac{1}{3}A_4 r + \ldots + \frac{1}{3}A_n r$

$= \frac{1}{3}\left(A_1 + A_2 + A_3 + A_4 + \ldots + A_n\right)r$

$= \frac{1}{3}$(Surface area of the sphere)r

$= \frac{1}{3} \times 4\pi r^2 \times r$

$= \frac{4}{3}\pi r^3$

$\therefore$ The volume, V, of a sphere in cubic units is given by

$V = \frac{4}{3}\pi r^3$

where r is the radius of the sphere.

Example 1

Find the surface area and the volume of the hemisphere with radius 5.

Solution

a) Surface area $= 4\pi r^2 \times 0{,}5$

$= 4(\pi)(5)^2 \times 0{,}5 = 157{,}1$ u^2

b) Volume $\frac{4}{3}\pi r^3 \times 0{,}5$ - hemisphere

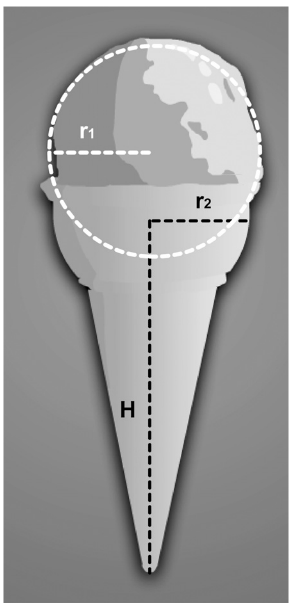
$= \frac{4}{3}(\pi)(5)^3 \times 0{,}5 = 231{,}8$ u^3

3. The radius r_1, of the ball of ice cream put into a cone is 4 cm.

The radius of the opening of the cone r_2, is 3,5 cm.

The depth of the cone, H, is 9 cm.

If the ice cream is pushed down, will the cone be able to take it?



Solution

Volume of the ball of ice cream $= \frac{4}{3}\pi(r_1)^3$

$= \frac{4}{3}\pi \cdot 4^3$

$= 268{,}08 \ldots \text{ cm}^3$

Capacity of the cone $= \frac{1}{3}\text{A H} = \frac{1}{3}\pi(r_2)^2.\text{H}$

$= \frac{1}{3}\pi \times 3{,}5^2 \times 9$

$= 346{,}36 \ldots \text{ cm}^3$

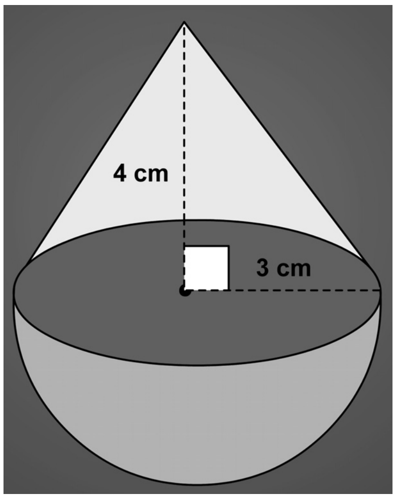
So yes, the cone will be big enough to take all the ice cream in the ball with space to spare.

Example

A child's toy is made from a cone with height 4 cm joined to a hemisphere with radius 3 cm.

a) Calculate the volume of the toy.

b) Calculate the total surface area.



Solution

a) Volume = vol of cone + vol of hemisphere

$= \frac{1}{3}\pi r^2.\text{H} + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{\pi}{3}(3^2)4 + \frac{2}{3}\pi(3)^3$

$= 12\pi + 18\pi$

$= 30\pi$

$= 94{,}3 \text{ cm}^3$

b) Total surface area = SA of hemisphere + SA of cone curved

$= \frac{1}{2}(4\pi r^2) + \pi rs$

$= 2\pi(3)^2 + \pi(3)(5)$

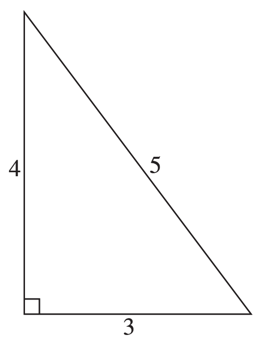
$= 18\pi + 15\pi$

$= 33\pi$

$= 103{,}7 \text{ cm}^2$

$s^2 = 16 + 9 = 25$

$\therefore s = 5$



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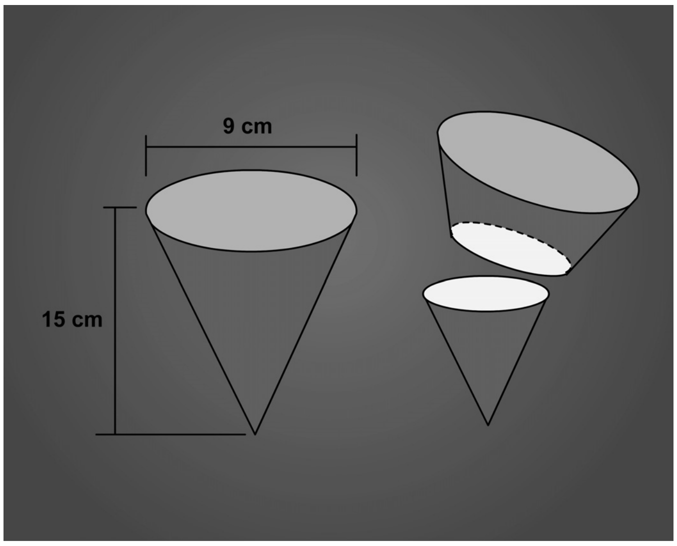
Example

Eddie cuts a large wooden cone into a smaller cone and a frustum.

The smaller cone has a diameter of 6 cm and a height of 10 cm.



a) Calculate the volume of the frustum.

b) Calculate the curved surface area of the frustum.

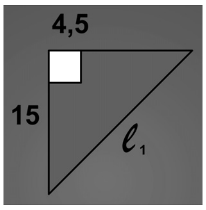
Solution

Volume of frustum = volume of big cone - volume of small cone

$= \left(\frac{1}{3}\right)(\pi)(4,5)^2(15) - \left(\frac{1}{3}\right)(\pi)(3)^2(10)$

$= 223,8 \text{ cm}^3$

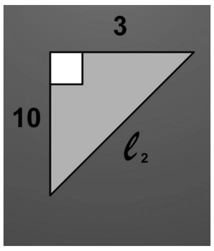
Curved area of big cone - curved surface area of small cone



Slant height of big cone

$s^2 = 15^2 + (4,5)^2$

$\therefore\ s_1 = 15,7$ u



Slant height of small cone

$(S_2)^2 = 10^2 + 3^2$

$= 109$

$S_2 = 10,4$

Curved surface area = π (4,5)(15,7) - π (3)(10,4) = 123,9 u^2

Activity 8

SUMMATIVE ASSESSMENT
INDIVIDUAL
DVD

1. The picture shows a model of a rural hut.

   Assume that the hut is hemispherical with a radius of 3 metres.

   Calculate:

   a) the volume of air inside the hut

   b) the outer surface area of the hut.

2. The South African Large Telescope (SALT) is situated in the Sutherland Observatory in the Western Cape. The telescope is the largest in the southern hemisphere. The housing of the telescope is in the form of a cylinder on top of which is a hemispherical geodesic dome. The radius is the cylindrical section is 13 m and its height is 17 m.

   Calculate:

   a) the outer surface area of the structure

   b) an estimate of the volume of air inside the structure.

3. The cone in the Great Zimbabwe Ruins is made of stone bricks, individually shaped from spars of granite. The measurement of each brick is roughly 35 cm by 12 cm.

   The diameter of the base of the cone is 5,5 m and its original height is estimated at 10 m.

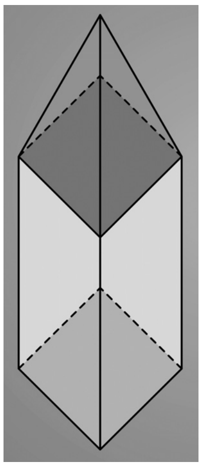
   Calculate, assuming the dimensions given:

   a) the volume of one brick

   b) the volume of the cone

   c) an estimate of the number of bricks in the cone.

4. The solid in the diagram is made up of a right prism with square base, and a right pyramid on top of the prism.



   The length of the prism is 12 cm, the side of the base is 6 cm and the height of the pyramid is 8 cm.

   Calculate:

   a) the slant height of a triangular face of the pyramid

   b) the area of one of the triangular faces

   c) the total surface area of the solid

   d) the volume of the solid.

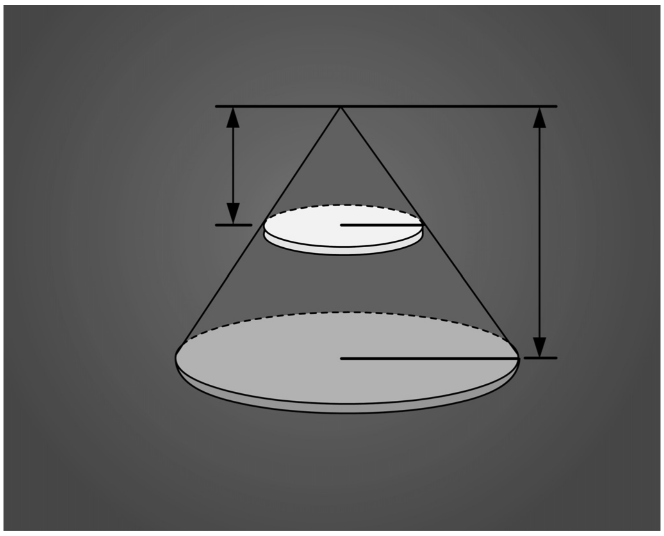
5. The average radius of the earth is 6 378 km.

   Approximately 71% of the surface of the earth is covered with water.

   Calculate the area of the earth's surface that is not covered by water in millions of square kilometres.

6. Rubber bungs are made by removing the tops of cones. Starting with a cone of radius 10 cm and height 16 cm, a rubber bung is made by cutting a cone of radius 5 cm and height 8 cm from the top.

   Find the volume and total surface area of the rubber bung.



7. A quadrant of a circle is cut out of paper.

   A cone is made by joining the edges OA and OB, with no overlaps.

   Calculate:

   a) the length of the arc AB

   b) the base radius of the cone

   c) the height of the cone

   d) the volume of the cone

ANALYTICAL GEOMETRY (1)

Lesson 27

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
- The gradient and inclination of a straight line
- The equation of a straight line

Overview

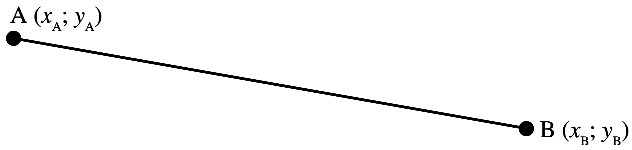
In this lesson you will:

- Review the distance formula, the mid-point and the gradient covered in Grade 10 in order to cope with the progression needed for this section of the curriculum.
- Use analytical geometry and properties of quadrilaterals to solve various problems.
- Use correct formulae, interpret questions and make the necessary equations.

Lesson

DVD

The distance formula:



To find the distance between two points A and B, we use the formula:

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

Examples

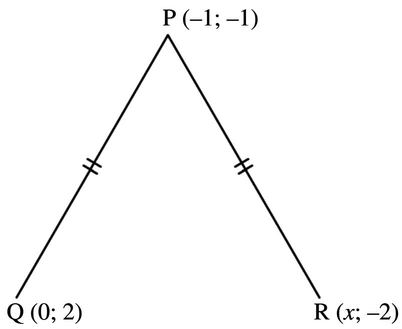
1.The distance between A(–7; y) and B(–3; 4) is $4\sqrt{5}$, find y

$$\begin{aligned}
AB &= 4\sqrt{5} \qquad \text{(given)} \\
\therefore\ AB^2 &= (4\sqrt{5})^2 \\
&= 16 \cdot 5 \\
&= 80 \\
\therefore\ (x_A - x_B)^2 + (y_A - y_B)^2 &= 80 \\
\therefore\ (-7 + 3)^2 + (y - 4)^2 &= 80 \\
\therefore\ (y - 4)^2 &= 80 - 16 \\
&= 64 \\
\therefore\ y - 4 &= 8 \\
\therefore\ y &= 4 \pm 8 \\
y = 12 \quad &\text{or} \quad y = -4
\end{aligned}$$

2. $P(-1; -1)$ is equidistant from $Q(0; 2)$ and $R(x; -2)$

Find the value of x

Draw a picture



Form an equation PQ = PR

So $PQ^2 = PR^2$

Distance formula:

$\therefore (x_P - x_Q)^2 + (y_P - y_Q)^2 = (x_P - x_R)^2 + (y_P - y_R)^2$

$\therefore (-1 - 0)^2 + (-1 - 2)^2 = (x + 1)^2 + (-2 + 1)^2$

$\therefore 1 + 9 = (x + 1)^2 + 1$

$\therefore (x + 1)^2 = 9$

$\therefore x + 1 = \pm 3$

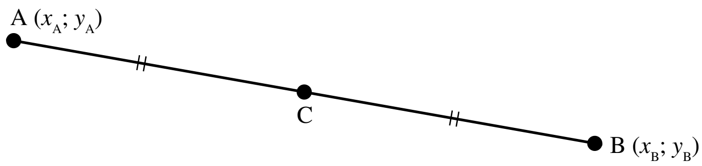
$\therefore x = -1 \pm 3$

$\therefore x = -4 \quad \text{or} \quad x = 2$

The mid-point of a line

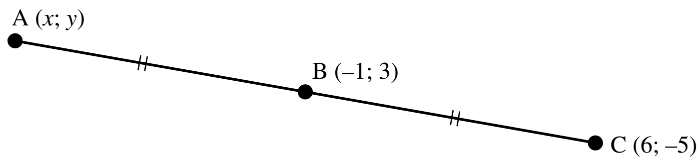
To determine the co-ordinates of the midpoint of a line segment AB:

$$x_C = \frac{x_A + x_B}{2} \qquad y_C = \frac{y_A + y_B}{2}$$



Example

B(–1; 3) is the mid-point of AC.



Find the co-ordinates of A if

$A(x; y)$ and $C(6; -5)$

Draw a picture

Form equations

$-1 = \frac{x + 6}{2}$ and $3 = \frac{y - 5}{2}$

$-2 = x + 6 \qquad 6 = y - 5$

$x = -8 \qquad y = 11$

$A(-8; 11)$

Activity 1a–e

BASELINE ASSESSMENT

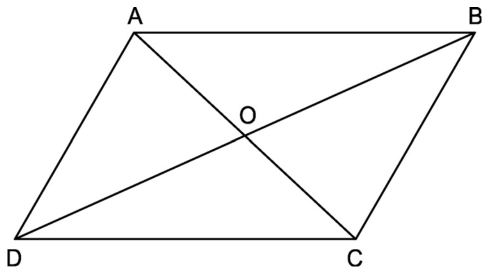
PAIRS

DVD

Working with the parallelogram

Diagonals bisect each other

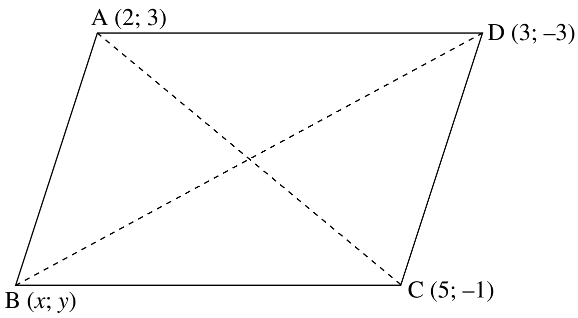
Show that the mid-point of AC is the same as the mid-point of BD.



Example

A(2; 3), C(5; –1), B(x; y) and D(3; –3) are the co-ordinates of the parallelogram ACBD, find the co-ordinates of B.

Gradient



Since ABCD is a $\|^m$, we can use the properties of $\|^m$ to help us find the co-ordinates of point B.

Using the fact that the diagonals bisect one another:

Midpoint of AC: $\left.\begin{array}{l} x_E = \dfrac{x_A + x_C}{2} = \dfrac{5+2}{2} = \dfrac{7}{2} \\ y_E = \dfrac{y_A + y_C}{2} = \dfrac{3-1}{2} = 1 \end{array}\right\} E\left(\dfrac{7}{2}; 1\right)$

Now midpoint BD:

$$\begin{array}{ll} x_E = \dfrac{x_B + x_D}{2} \text{ and} & y_E = \dfrac{y_B + y_D}{2} \\ \dfrac{7}{2} = \dfrac{x+3}{2} & 1 = \dfrac{y-3}{2} \\ \therefore x = 7 - 3 & \therefore y - 3 = 2 \\ x = 4 & y = 5 \end{array}$$

$$\therefore E = (4; 5)$$

Using the fact that the gradients of sides are equal:

$m_{AB} = m_{DC}$

$\therefore \triangle y_{AB} = \triangle y_{DC}$ and $\triangle x_{AB} = \triangle x_{DC}$

$\therefore y - 3 = -1 - (-3)$ $\quad$ $\therefore x - 2 = 5 - 3$

$\therefore y - 3 = 2)$ $\quad$ $\therefore x = 2 + 2$

$y = 5$ $\quad$ $x = 4$

$$\therefore E = (4; 5)$$

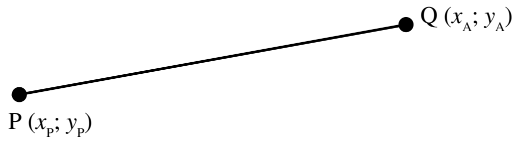
We can use BC = AD, and the equation of BC, to find the co-ordinates of B.

We could also use $m_{BC} = m_{AD}$ in the same way as above.

The gradient of a line segment

To determine the gradient (slope) of a line PQ, we use

$$m_{PQ} = \frac{\triangle y}{\triangle x} = \frac{y_Q - y_P}{x_Q - x_P}$$



If two lines are parallel, then they have equal gradients

If two lines are perpendicular, then the product of their gradients is –1

If a line is horizontal, then $\triangle y = 0$: So m = 0

If a line is vertical, then $\triangle x = 0$: So m = α

3 points A, B and C are said to be collinear if $m_{AB} = m_{BC}$

Examples

If A(1; 4), B(–3; 2), C(–1; –1) and D(x; 0) are four points in the Cartesian plane, find the value of x if:

(a) AB || CD

(b) AB ⊥ BD

(c) BC and D are collinear.

1. a) AB || CD Make the equation

$$m_{AB} = m_{CD}$$
$$\frac{2-4}{-3-1} = \frac{0+1}{x+1}$$
$$\frac{-2}{-4} = \frac{1}{x+1}$$
$$\frac{1}{2} = \frac{1}{x+1} \text{ cross multiply}$$
$$x + 1 = 2$$
$$x = 1$$

b) AB ⊥ BD

$$m_{AB} \times m_{BD} = -1$$
$$\therefore \frac{1}{2} \times \frac{2-0}{-3-x} = -1$$
$$\frac{1}{2} \times \frac{2}{-3-x} = -1$$
$$\therefore \frac{1}{x+3} = 1$$
$$\therefore 1 = 3 + x$$
$$x = -2$$

c) B; C and D are collinear

$m_{BC} = m_{CD}$

$\frac{2 + 1}{-3 + 1} = \frac{0 + 1}{x + 1}$

$-\frac{3}{2} = \frac{1}{x + 1}$

$3(x + 1) = -2$

$3x + 3 = -2$

$3x = -5$

$x = \frac{-5}{3}$

Activities 2 – 4

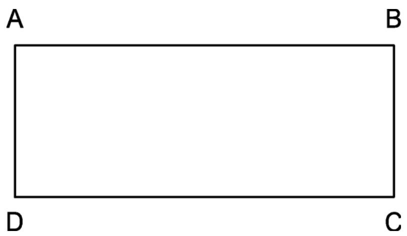
FORMATIVE ASSESSMENT

PAIRS

DVD

How to prove a quadrilateral is a rectangle

1. Prove ABCD is a parallelogram



So mid-point of AC = mid-point BD

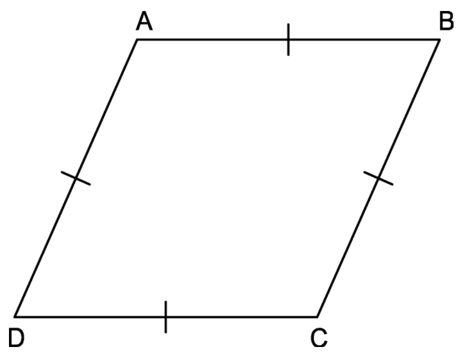
2. Then prove that there is one right angle.

Perhaps $m_{AD} \times m_{AB} = -1$

Then $\hat{A} = 90°$

How to prove a quadrilateral is a rhombus, or prove that the diagonals bisect each other at 90°

Prove all four sides are equal.



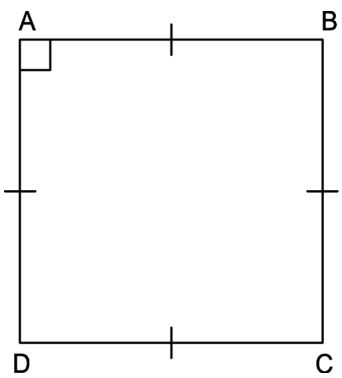
How to prove a quadrilateral is a square

Prove all four sides are equal

Prove there is one right angle

perhaps $m_{AB} \times m_{BC} = -1$

so $\hat{B} = 90°$



BASELINE ASSESSMENT

PAIRS

Activity 5

1. In each case determine the value of x and y.

   a) The distance of $(2; y)$ from the origin is $\sqrt{40}$.

   b) $(2; –3)$ is equidistant from $(–3; 2)$ and $(x; –8)$.

   c) $(x; y)$ is the mid-point of the line segment joining $(–1; 3)$ and $(7; 1)$.

   d) $(1;-2)$ is the centre of the circle passing through $(5; 1)$ and $(–2; y)$.

2. If A(3; 1), B–5; 7), C(11; –5) and D(x; y) are the co-ordinates of parallelogram ABCD, find the co-ordinates of D.

3. Prove that K(4; 4), L(5; –1) and M(–6; 2) are the vertices of a right angled triangle and hence decide which angle is right angled.

4. Find p and q if P(–3; 2) is the midpoint of the line joining A(p; q) and B(–1; 5).

5. The vertices of ΔDEF are D(2; 3), E(–3; –1) and F(6; –2). Show that it is a right angle triangle. If P and Q are the midpoints of DE and EF respectively, prove also that PQ||EF and that PQ = $\frac{1}{2}$EF.

6. Prove that the figure bounded by the lines $y = 2x + 1$; $y – 3x = 6$; $2y + 6x = 7$; $y = 3x + 1$ is a trapezium. Find the coordinates of its vertices.

ANALYTICAL GEOMETRY (2)

Lesson 28

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
- The gradient and inclination of a straight line
- The equation of a straight line

Overview

In this lesson you will:

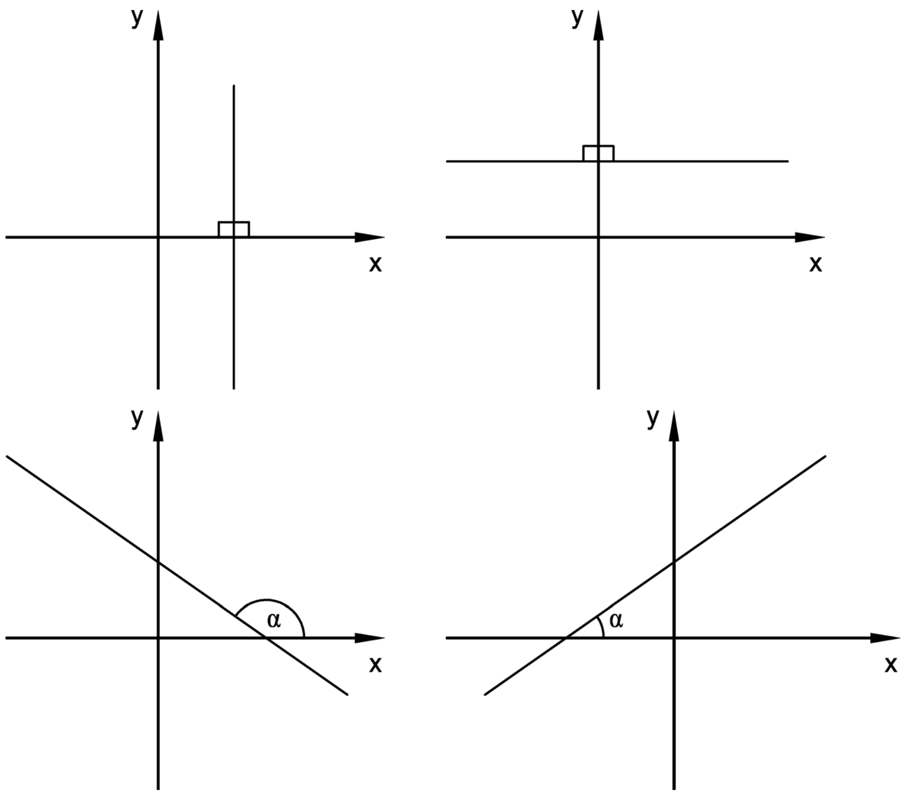
- Discover what is meant by inclination.
- Use trigonometry to find the inclination of a straight line.
- Find the angle between two straight lines.
- Use analytical methods to find the three angles of a triangle.

Lesson

The inclination of a straight line

Definition: The angle formed by a straight line and the positive direction of the *x*-axis

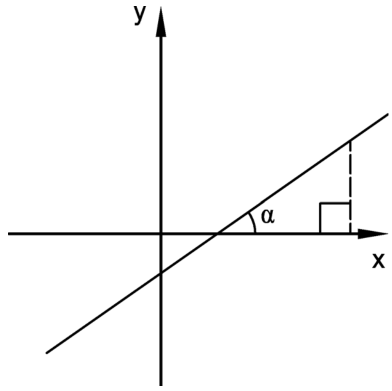
4 OPTIONS



To find this angle we need the concept of a gradient and link it directly to trigonometry.

$m = \dfrac{\text{change in } \triangle y}{\text{change in } \triangle x}$

and $\tan \alpha = \dfrac{y}{x}$

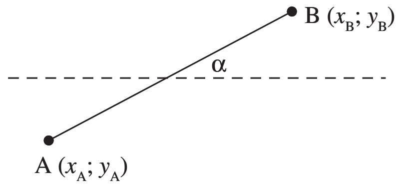


$\therefore$ For any line segment AB:

$\tan \alpha = m_{AB}$

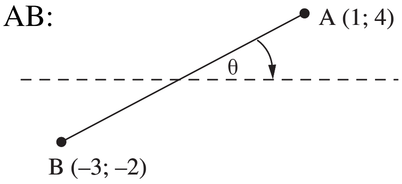
$\therefore \tan \alpha = \dfrac{y_B - y_A}{x_B - x_A}$

So $\alpha = \tan^{-1}(m_{AB})$



Examples

1. We introduce any horizontal line through AB:



Then $\tan \theta = m_{AB}$

$= \dfrac{4+2}{1+3}$

$= \dfrac{6}{4}$

$= \dfrac{3}{2}$

$\therefore \theta = \tan^{-1}\left(\dfrac{3}{2}\right)$

$= 56{,}3°$

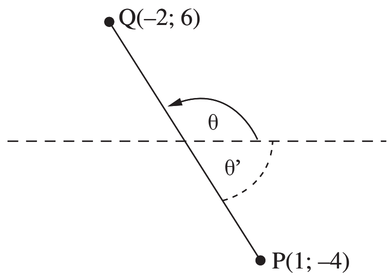
2. $\tan \theta = m_{PQ}$

$= \dfrac{6+4}{-2-1}$

$= -\dfrac{10}{3}$

$\therefore \theta = \tan^{-1}\left(-\dfrac{10}{3}\right)$

$= -73{,}3°$



Notice that when $\tan \theta < 0$, we are working with the negative angle θ! So to get to θ, we need to add a period of tan, which is 180°.

So $\theta = \theta' + 180° = -73{,}3° + 180° = 106{,}7°$.

3. Find the angle $2y + x = 4$ makes with the positive direction of the x-axis

$2y = -x + 4$

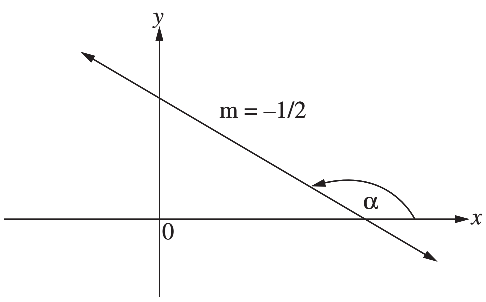
$y = -\dfrac{1}{2}x + 2$

$m = -\dfrac{1}{2}$

$\tan \alpha = -\dfrac{1}{2}$

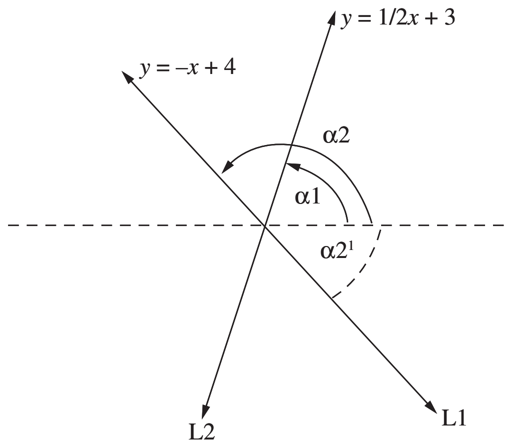
$\alpha = 180° - 26{,}6°$

$= 153{,}4°$



4. Find the size of the angles between the following lines

1. $y = -x + 4$ and $y = \frac{1}{2}x + 3$
2. $y = -3x - 4$ and $y = -x$
3. $y = 8x$ and $y = 2x + 3$



1. For L_1: $\tan \alpha_2 < 0$

   So $\alpha_2 = \tan^{-1}(-1) + 180°$

   $= -45° + 180°$

   $= 135°$

   $\alpha_2' = 45°$

   For L_2: $\alpha_1 \tan^{-1}\left(\frac{1}{2}\right) + 26{,}6°$

   The acute angle between the lines will be: $45° + 26{,}6° = 71{,}6°$

   The obtuse angle between them is: $180° - 71{,}6° = 108{,}4°$

2. For L_1: $m = -1$

   $\theta_1 = \tan^{-1}(-1) + 180°$

   $= -45° + 180°$

   $\therefore \theta_1 = 135°$

   and $\theta_1' = 45°$

   For L_2: $m = -3$

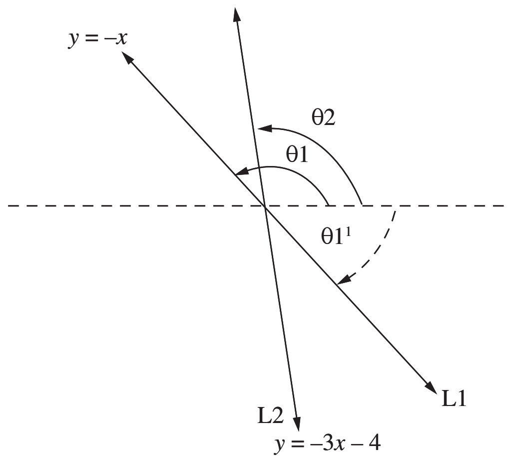
   $\theta_2 = \tan^{-1}(-3) + 180°$

   $= -71{,}6° + 180°$

   $\theta_2 = 108{,}4°$

   Thus: Obtuse angle: $108{,}4° + 45 = 153{,}4°$

   Acute angle: $26{,}6°$



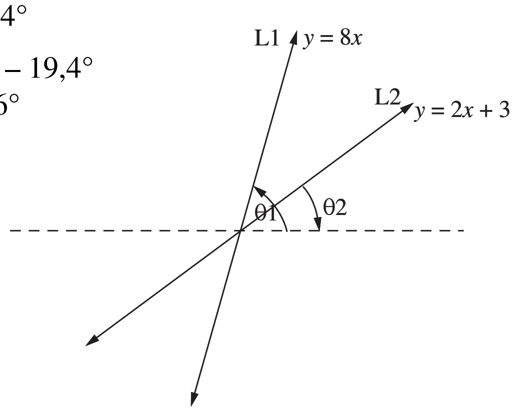
To use this concept effectively, always draw a diagram

3. L_1: $m = 8$: $- \theta_1 = \tan^{-1}(8) = 82{,}8°$

   L_2: $m = 2$: $- \theta_2 = \tan^{-1}(2) = 63{,}4°$

   $\therefore \angle$ between lines (acute) $= 82{,}8 - 63{,}4°$
   $= 19{,}4°$

   The obtuse angle will be $= 180° - 19{,}4°$
   $= 160{,}6°$



FORMATIVE ASSESSMENT

PAIRS

DVD

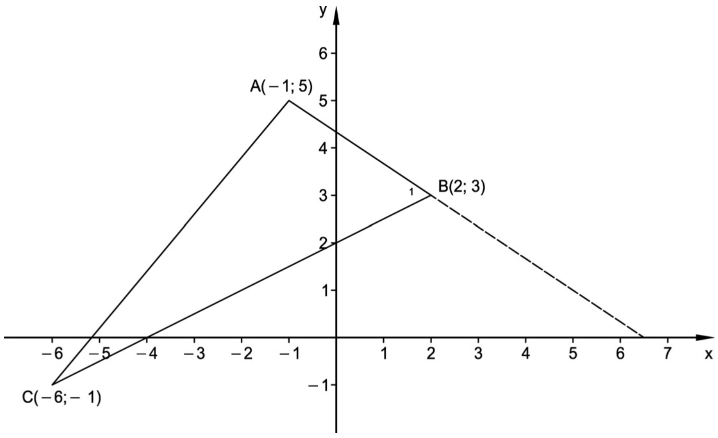
Activities 1–4

5. A(–1; 5) B(2; 3) and C(–6; –1) are co-ordinates of the vertices of △ABC

Find the size of the angles.

Draw a picture

Tip: Find α, β and θ the use geometry.



$m_{AC} = \frac{6}{5}$ $\quad$ $m_{BC} = \frac{4}{8}$ $\quad$ $m_{AB} = \frac{2}{-3}$

$\tan\alpha = \frac{6}{5}$ $\quad$ $\tan\beta = \frac{1}{2}$ $\quad$ $\tan\theta = -\frac{2}{3}$

$\alpha = 50{,}2°$ $\quad$ $\beta = 26{,}6°$ $\quad$ $\theta = 146{,}3°$

By geometry $\hat{C} = \alpha - \beta$

$\hat{C} = 23{,}6°$

$\hat{A} = \theta - \alpha$

$\hat{A} = 96{,}1°$

$\hat{B}_1 = 60{,}3$ (<'s in △)

Let's find the inclination of each side

First:

AC: $m = \frac{5+1}{-1+6} = \frac{6}{5}$

$\therefore \theta = \tan^{-1}\left(\frac{6}{5}\right) = 50{,}2°$

AB: $m = \frac{5-3}{-1-2} = -\frac{2}{3}$

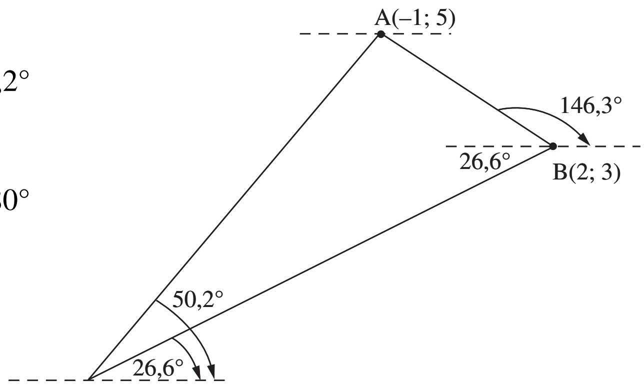
$\therefore \beta = \tan^{-1}\left(-\frac{2}{3}\right) + 180°$

$= 146{,}3°$

BC: $m = \frac{3+1}{2+6} = \frac{1}{2}$

$\alpha = \tan^{-1}\left(-\frac{1}{2}\right)$

$= 26{,}6°$



$\hat{C} = \theta - \alpha = 50{,}2°\ 26{,}6°$

$\hat{C} = 23{,}6°$

$\hat{B} = (180° - 146{,}3°) + 26{,}6° = 60{,}3°$

$\hat{A} = 180° - 23{,}6° - 60{,}3°$ $\quad$ Sum angles in triangle

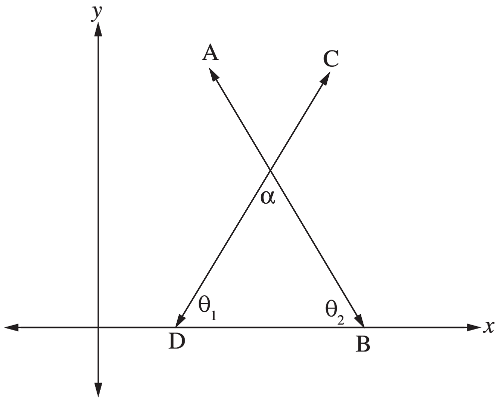
$= 96{,}1°$

The angle between two lines AB and BC

$\alpha = \theta_2 - \theta_1$

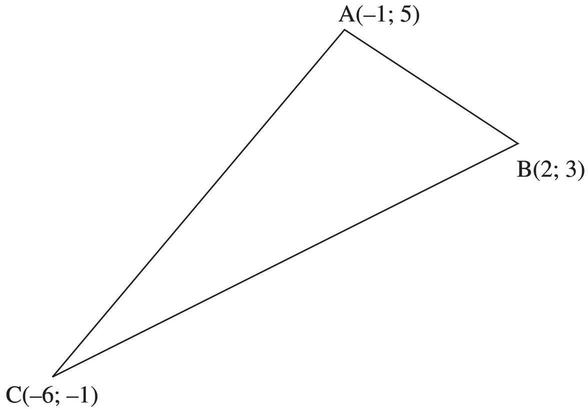
The acute angle between AB and CD:

$\tan \ = \dfrac{m_{AB} - m_{CD}}{1 + m_{AB} \cdot m_{CD}}$



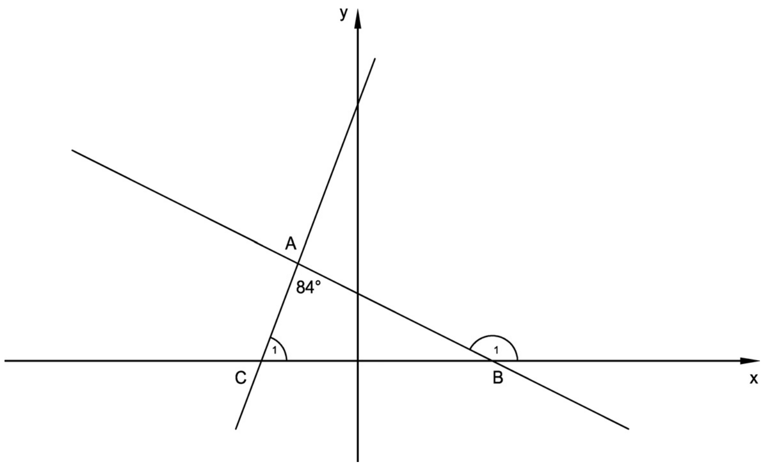
Applying this to:

$$m_{AC} = \frac{6}{5}$$
$$m_{AB} = -\frac{2}{3}$$
$$m_{BC} = \frac{1}{2}$$
$$\tan \hat{A} = \frac{m_{AB} - m_{AC}}{1 + m_{AB} \cdot m_{AC}}$$
$$= \frac{-\frac{2}{3} - \frac{6}{5}}{1 + \left(-\frac{2}{3}\right)\left(\frac{6}{5}\right)}$$
$$= -\frac{28}{3}$$
$$\therefore \hat{A} = \tan^{-1}\left(-\frac{28}{3}\right) + 180°$$
$$= 96,1°$$



6. Find the gradient of AC if it is given that $\tan \hat{B}_1 = -\frac{1}{2}$

$\hat{B}_1 = \hat{C}_1 + 84°$ Exterior ∟ of △



$\hat{B}_1 = 180° - 26,6°$ $\hat{C}_1 = 153,4° - 84°$

$\hat{B}_1 = 153,4°$ $\hat{C}_1 = 69,4°$

Gradient of AC = tan 69,4° = 2,7

FORMATIVE ASSESSMENT

PAIRS

Activities 5–9

1. Calculate the inclination of AB if

   a) A(–3; 0) and B(6; 8) b) A(–4; $\sqrt{2}$) and B(7; 1)

   c) A($4\sqrt{3}$; –1) and B(2; $\sqrt{3}$) d) A(3; 4) and B(3; –6)

   e) A(1; –1) and B(6; –1)

2. Calculate the inclination of the following straight lines

   a) $3y + x = 4$ b) $2y - x = 3$

   c) $4x + y = 1$ d) $x = \frac{1}{2}y$

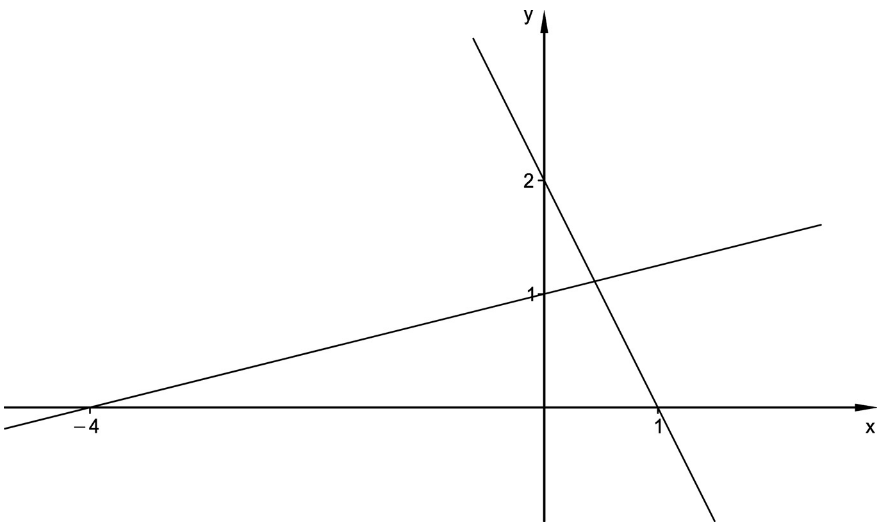
3. Find $D\hat{E}F$ if D(5; –2), E(–3; 7) and F(2; 5).

4. $y = -\frac{1}{2}x$ and $y = 3x + 3$

   Intersect at one place.

   Calculate the possible sizes of the angles at the point of intersection.

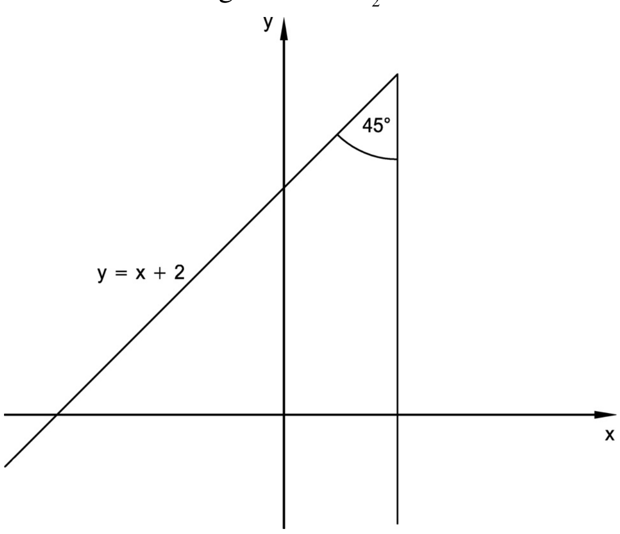
5. Calculate α and β



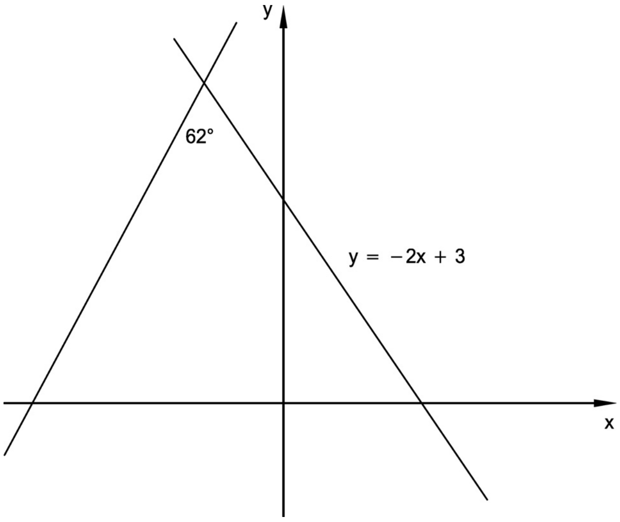
SUMMATIVE ASSESSMENT

INDIVIDUAL

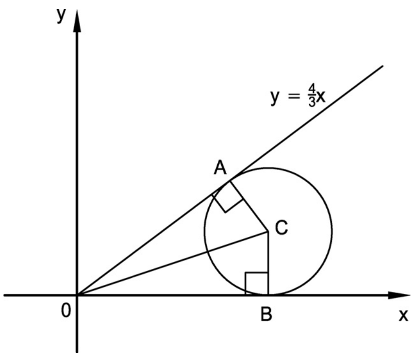
6. $\hat{A} = 45°$ Find the gradient of ℓ_2



7. Find the gradient of AB



8. Calculate the gradient of OC



9. P(–1; 4) Q(2; 2) and
R(–6; –1) are co-ordinates of △PQR, find the angles of the triangle.

ANALYTICAL GEOMETRY (3)

Lesson 29

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
- The gradient and inclination of a straight line
- The equation of a straight line

Overview

In this lesson you will:

- Use a formula to find the equation of a straight line.
- Find equations of parallel and perpendicular lines.
- Find the co-ordinates of the point of intersection of two lines.

Lesson

L C DVD

Equations of straight lines (Used to find the equations of lines)

Here we have four cases:

1. If the gradient $\mathbf{m_{AB}}$ and the ***y*-intercept** c of the line are given:

   ⇒ We use $y = \mathrm{m}_{AB}\, x + c$ (**the gradient/*y*-int formula**)

2. If the gradient $\mathbf{m_{AB}}$ and another point **C (*x*$_C$; *y*$_C$)** on the line are given:

   ⇒ We use $(y - y_C) = \mathrm{m}_{AB}\, x - x_C$ (**the point gradient formula**)

3. If **two points A(x_A ; y_A) and B(x_A ; y_B)** on the line are given:

   ⇒ We use $\frac{y - y_B}{x - x_B} = \frac{y_{AB} - y_B}{x_A - x_B}$ (**the two point formula**)

4. If the ***x*-intercept** and ***y-intercept*** are given:

   ⇒ We use $\frac{x}{x - \text{int}} + \frac{y}{y - \text{int}} = 1$ (**the dual intercept formula**)

Examples

1. Find the equation of the line passing through A (–1; –4) with a gradient of –2.

Solution

Here we have the gradient and a point, so we use our point gradient formula:

Now simply substitute

$y + 4 = -2(x + 1)$

$y + 4 = -2x - 2$

$\frac{y - y_A}{x - x_A} = \mathrm{m}$

$\therefore\ y + 4 = -2(x + 1)$

$y = -2x - 6$

2. Find the equation of the line AB if A(6; –2) and B(–3; 1)

Solution

Here we have two points:

So $\frac{y - y_A}{x - x_A} = \frac{y_A - y_B}{x_A - x_B}$

$\therefore \frac{y + 2}{x - 6} = \frac{-2 - 1}{6 + 3} = -\frac{1}{3}$

$\therefore 3(y + 2) = -(x - 6)$

$\therefore 3y + 6 = -x + 6$

$\therefore 3y = -x$

3. Find the equation of the line passing through A(2; –4) parallel to the line passing through B(1; 1) and C (–3; –2)

Solution

$m_{BC} = \frac{-2 - 1}{-3 - 1} = \frac{3}{4}$

Now we have a gradient and a point, so

$\frac{y - y_A}{x - x_A} = m_{BC}$

$y - (-4) = \frac{3}{4}(x - 2)$

$y + 4 = \frac{3}{4}(x - 2)$

$4y + 16 = 3x - 6$

$4y = 3x - 22$

Remember: if we write an equation in the form $y = mx + c$ then m is the gradient.

4. Find the equation of the line parallel to $2y + 4x = 5$ passing through the point P(–1; –6)

Solution

We need a gradient and a point P(–1; –6)

Gradient $2y = -4x + 5$

$y = -2x + \frac{5}{2}$

$m = -2$

We now have a gradient, and a point, so

$\frac{y - y_P}{x - x_P} = m$

Now substitute: $y + 6 = -2(x + 1)$

$y + 6 = -2x - 2$

$y = -2x - 8$

Remember: If AB $\perp$ DE then $m_{AB} \times m_{DE} = -1$

5. Find the equations of the line perpendicular to $4y + x = -8$ through the point Q(–1; –3)

Solution

We need the gradient

$4y = -x - 8$

$y = -\frac{1}{4}x - 2$

$m = -\frac{1}{4}$ So the gradient we want is 4

m = 4 and Q(–1; –3)

So $\frac{y - y_Q}{x - x_Q} = m$

Equation $y + 3 = 4(x + 1)$

$y + 3 = 4x + 4$

$y = 4x + 1$

6. Find the equation of the line perpendicular to the line joining A(–3; –4) and B(2; 6) and passing through C(–5; 1)

Solution

Gradient AB $= \frac{10}{5} = 2$

Our new line is perpendicular to AB so $m = -\frac{1}{2}$

So $\frac{y - y_C}{x - x_C} = m$

C(–5; 1)

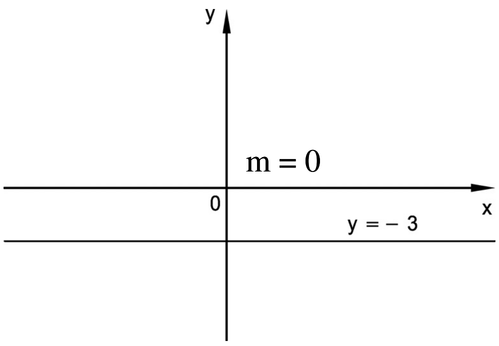
Equation $y - 1 = -\frac{1}{2}(x + 5)$

$2y - 2 = -(x + 5)$

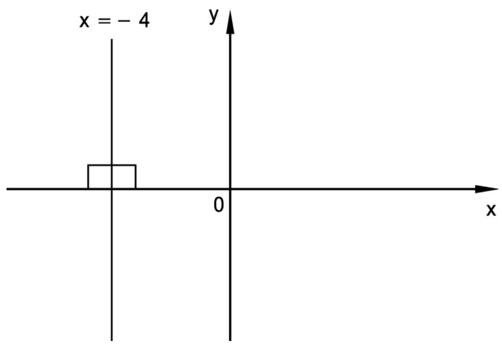
$2y - 2 = -x - 5$

$2y = -x - 3$

REMEMBER THOSE IMPORTANT LINES



$y = -3$



m is undefined
$x = -4$

7. (a) Find the equation of the line parallel to the x-axis through the point (4; 2)

Solution m = 0 so $y = 2$

(b) Find the equation of the line through the points (3; 1) and (3; –4)

Solution $m = -\frac{5}{0}$ m is undefined so $x = 3$

Activity

SUMMATIVE ASSESSMENT

INDIVIDUAL

1. The points A(3; 1) B(1; –2) and C(2; 3) are points in the Cartesian plane.
   (a) If CD || AB with D($-\frac{1}{3}$; t + 1) determine the values of t.
   (b) If B, A and E(r; 4) are collinear, determine the value of r.
2. The equation of the line PQ is $(3 - 2k)x = (k + 1)y = 12$

   Calculate the numerical value of k in each case, if line PQ
   a) is parallel to the line $y = 4x + 7$
   b) is perpendicular to the line through points (1; 3) and (–2; 4)
   c) passes through the point (–3; 4)
   d) is parallel to the x-axis.
   e) is parallel to the y-axis.
3. Find the equation of the straight lines
   a) Parallel to the line $y + 8 = 0$ and 6 units from (2; 1).
   b) Perpendicular to the line $y - 2 = 0$ and 4 units from (–1; 7)

Lesson 30

ANALYTICAL GEOMETRY (4)

(Continuation of Lesson 29)

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
- The gradient and inclination of a straight line
- The equation of a straight line

Overview

In this lesson *y*ou will:

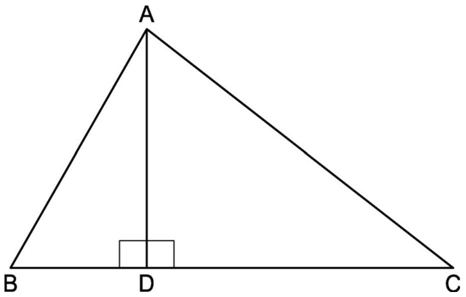
- Use a formula to find the equation of a straight line.
- Find equations of parallel and perpendicular lines.
- Find the co-ordinates of the point of intersection of two lines.

LC DVD

Lesson

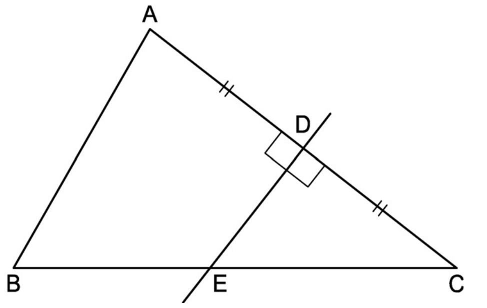
Important Lines

Altitude



AD is an altitude of △ABC. From the angle perpendicular to the opposite side.

Perpendicular Bisector

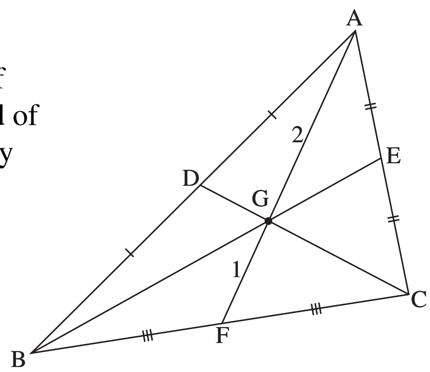


DE is the perpendicular bisector of AC

Medians and the Centroid

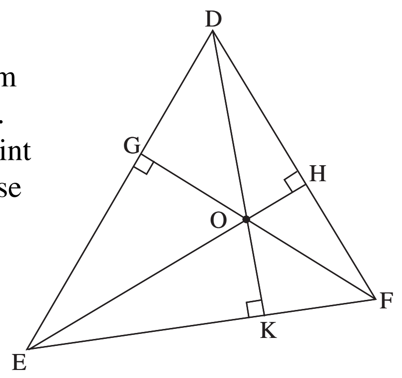
AF, BE and CD are the mediums of △ABC.

The medians run from vertex to the midpoint of the opposite side. Point G is called the **centroid** of △ABC, and this is also the point of concurrency of the medians.

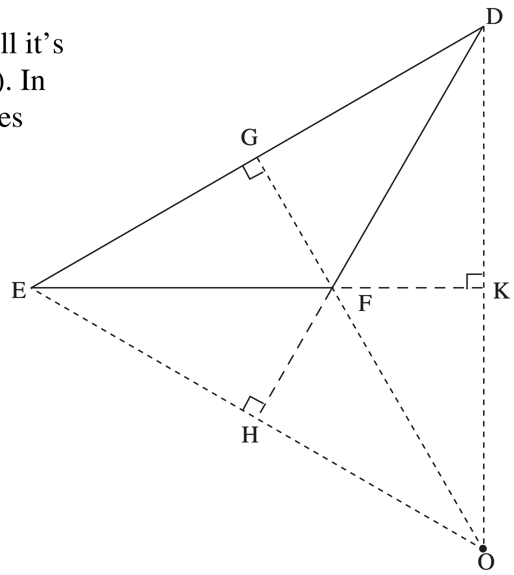


The Orthocentre and the Altitudes

The altitudes DK, FG and EH are drawn from the vertex perpendicular to the opposite side. (They don't necessary go through the midpoint of the side). The point of concurrency of these altitudes is called the **orthocentre**.

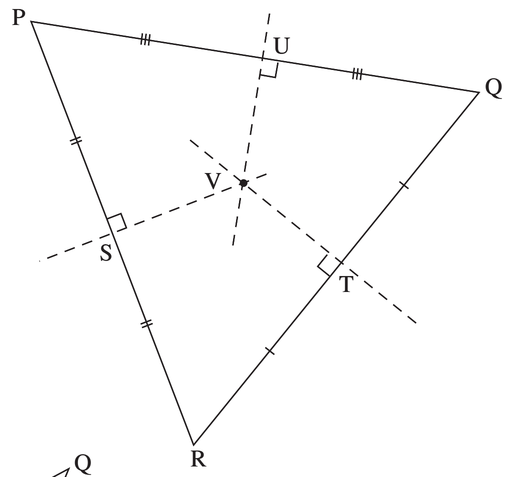


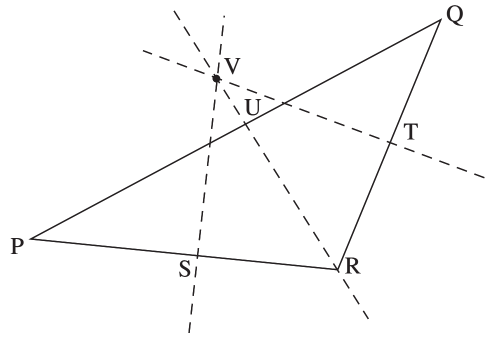
The orthocentre lies inside the triangle. All it's angles are smaller than 90° (acute angled). In obtuse angled triangles, the orthocentre lies outside the triangle.



The Circumcentre and the Perpendicular bisectors of the sides

△PQR has a circumference at V, where the perpendicular bisectors of the sides are concurrent. O is the centre of the circle that can be drawn through P, Q and R. In acute angled triangles the circumcentre lies inside the triangle. In obtuse angled triangles the circumcentre lies outside the triangle. Note that perpendicular bisectors of the sides, does not necessary pass through the vertices of the triangle.



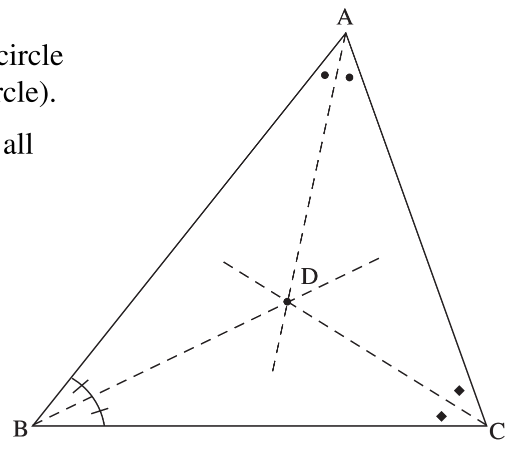


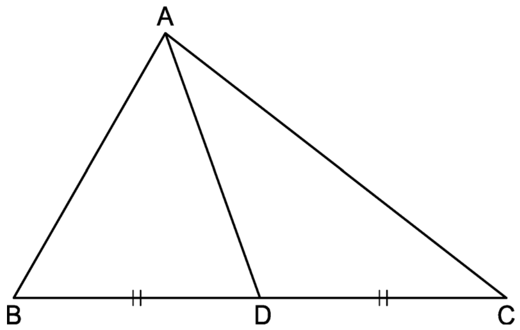
The Incentre and the Bisectors of the interior angles of a triangle

The bisectors of the angles A, B and C are concurrent at the incentre D of the triangle.

The incentre gives us the midpoint of the circle that can be drawn inside the triangle (incircle).

Incentres are located inside the triangle at all times.





AD is a median

From the angle to the middle of the opposite side

INVESTIGATION FOR ALL OF YOU. When will the altitude, perpendicular bisector and median be the same line.

Example 1

A(–3 ; 3) B(1 ; –1) and C(–5 ; –3) are the co–ordinates of △ABC

a) Find the equation of the altitude from A to BC

Solution

Plot the points roughly.

We want the equation of altitude AD.

We need a point and the gradient.

Point A(–3 ; 3) Gradient?

AD ⊥ CB

m_{CB} is $\frac{-1+3}{1+5} = \frac{2}{6} = \frac{1}{3}$

So m_{AD} –3

A(–3 ; +3)

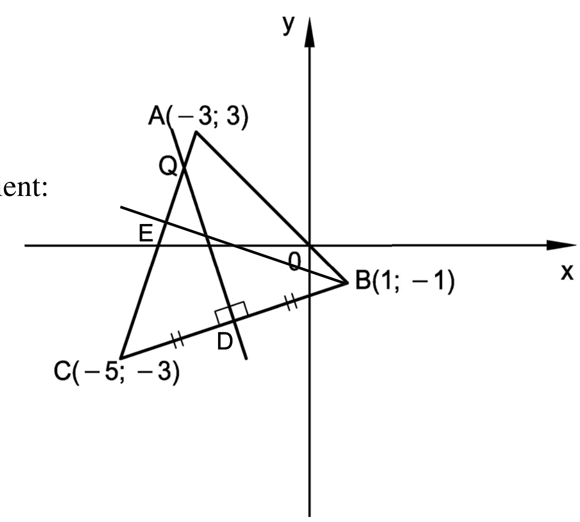
Equation is So point gradient:

$\frac{y - y_A}{x - x_A} = m$

$y - 3 = -3(x + 3)$

$y - 3 = -3x - 9$

$y = -3x - 6$



b) Find the equation of the median from B to AC.

Solution

To find the median, we need the midpoint of AC

$E = \left(\frac{-3-5}{2} ; \frac{3-3}{2}\right) \quad = (-4 ; 0)$

So equation of median: (2 points given)

$\frac{y-y_B}{x-x_B} = \frac{y_B - y_E}{x_B - x_E}$

$\therefore \frac{y+1}{x-1} = \frac{-1}{1+4} = -\frac{1}{5}$

$\therefore 5y + 5 = -x + 1$

$5y = -x - 4$

c) Find the equation of the perpendicular bisector of BC

Solution

We need the gradient of BC:

$m_{BC} = +\frac{1}{3} \rightarrow m_{\perp} = -3$

We need the midpoint of BC:

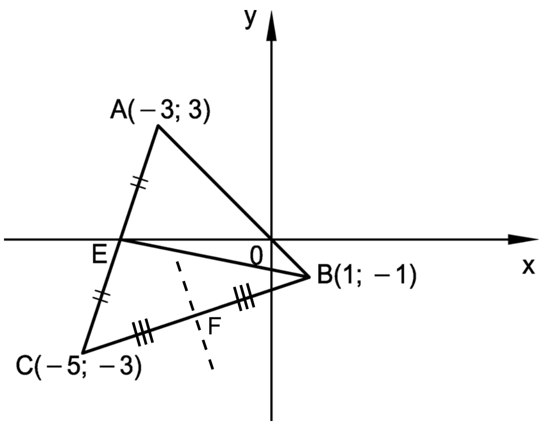
$F\ \frac{-5+1}{2}; \frac{-3-1}{2} = F(-2; -2)$ pt$(-2 ; -2)$

Equation of   bisector

$\frac{y+2}{x+2}$

$\therefore y + 2 = -3x - 6$

$\therefore y = -3x - 8$



Activities 1 – 4

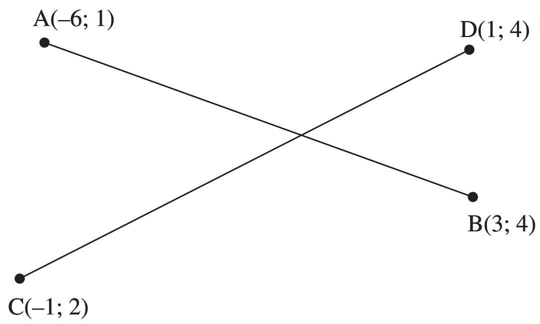
Points of Intersection

Example

A(–6 ; 1)   B(3 ; 4)   C(–1 ; 2) and D(1 ; 4) are four points.

Find the co–ordinates of the point of intersection of lines AB and CD.

Solution



Plan: Equation of AB
Equation of CD
Solve simultaneously

$m_{AB} = \frac{3}{9} = \frac{1}{3}$

$Eq_{AB}: \frac{y-4}{x-3} = \frac{1}{3}$

$3y - 12 = x - 3$

$3y = x + 9$

$m_{CD} = \frac{2}{2} = 1$

$Eq_{CD}: \frac{y-4}{x-1} = 1$

$y - 4 = x - 1$

$\therefore y = x + 3$

Solve simultaneously

Substitute 2 into 1

$3(x + 3) = x + 9$

$3x + 9 = x + 9$

$2x = 0$

$x = 0$

$y = 3$

Point of interception (0 ; 3)

How do we know if a point is on a line?

Is the point (9 ; –3) on the line $2y + x = 3$?

Plan: Make $x = 0$ and $y = -3$ and see whether the equation is true

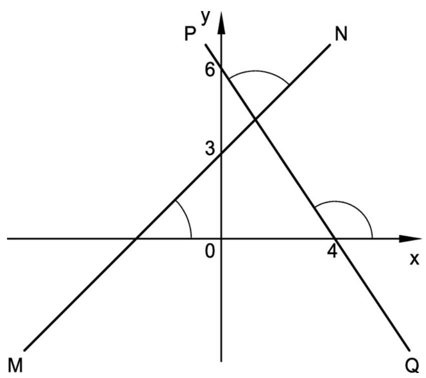
LHS $\quad 2y + x = 2(-3) + 9 = 3 = \text{RHS}$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activities 5–7

1.



Determine the size of

a) α b) β c) θ

2. M is a point t units from the origin O(0 ; 0).

   Calculate the co-ordinates of M in terms of t if the line $x + \sqrt{3y} = 2t$ passes through M.

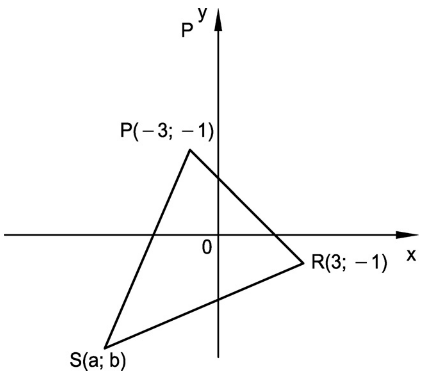
3. The vertices of $\triangle$ABE are A(0 ; 4) : B(5 ; 3) and E(2 ; 1)

   a) Prove that $\hat{E} = 90°$

   b) If ABCD is a rhombus with diagonals AC and BD intersecting at E, determine the co-ordinates of C and D.

   c) Prove ABCD is a square.

4. $\triangle$PRS has vertices P(–1 ; 3) R(3 ; –1) 5(a ; b)



a) Find T, the mid–point of PR.

b) If the perpendicular bisector of PR passes through S, show that $a = b$.

c) If the area of $\triangle$PRS is $12u^2$, find the co–ordinates of S.

d) If Q is (4 ; 4) show that PQRS is a rhombus.

5. a) M(a ; 2) is the mid point of A(–2 ; 5) and B(8 ; –1).

Find the value of a.

b) Find e if the distance between the point (0 ; –4) and (e ; 0) is 5.

c) Given the point A(1 ; 3) B(3 ; 2) and C(–1 ; –1) find

i) the equation of the straight line through C parallel to AB.

ii) the equation of the straight line through B perpendicular to AB.

iii) The point of intersection of these lines.

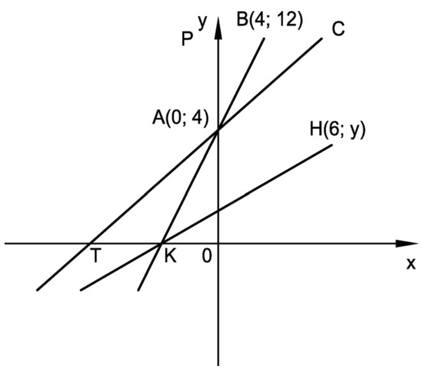
6. ABCD is a trapezium with co-ordinates A(–4 ; 3) B(x ; 6) C(4 ; y) and D(–2 ; –1) where $x < 4$.

If AD || BC and BC = 2AD

a) Find x and y.

b) Find the co-ordinates of E, the point of intersection of BA and CD.

7.



A(0 ; 4) B(4 ; 12) are the points seen above. Line TAC has a gradient of $\frac{1}{3}$

a) Find $C\hat{T}X$

b) Calculate $B\hat{A}C$

c) If H is the point (6 ; y) and $H\hat{K}X = 30°$, calculate the value of y.

(Leave answer in surd form)

Lesson 31

GEOMETRY

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and movement
Assessment Standard
a) Investigate necessary and sufficient for polygons to be similar.
b) Prove and use (accepting result proved in earlier grades):
- That the line drawn parallel to one of a triangle divides the other two sides proportionally. (the mid-point theorem is a special case of this theorem).
- The equiangular triangles are similar.
- That triangles with sides in proportion are similar.
- The Pythagorean theorem by similar triangles.

Overview

In this lesson you will:

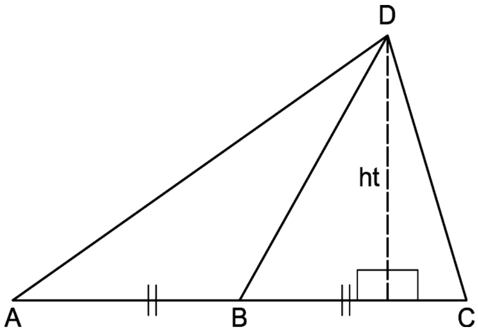
- Review the area of a triangle.
- Revise the concept of ratio learnt in previous grades.
- Informally prove and use the mid-point theorem.
- Prove and use the proportional intercept theorem.

Lesson

DVD

First let's look at area of triangles.

Area $\triangle$ABD = Area $\triangle$BCD

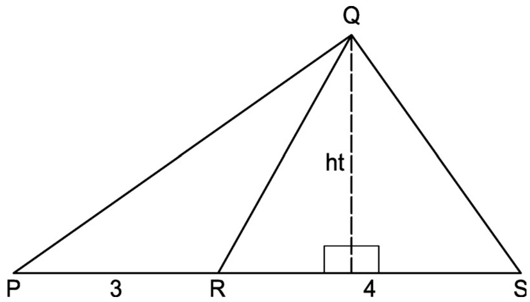


Why?

The height (altitude) is the same, and they have the same base lengths. In short we say: Same apex, same base.

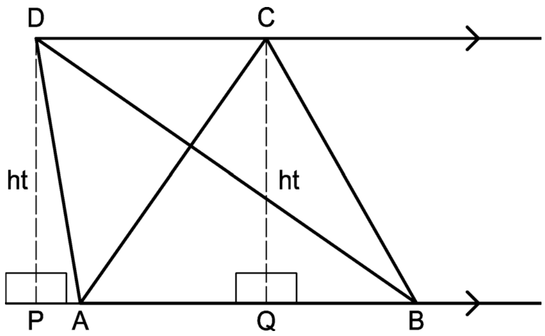
Some interesting results

(a)



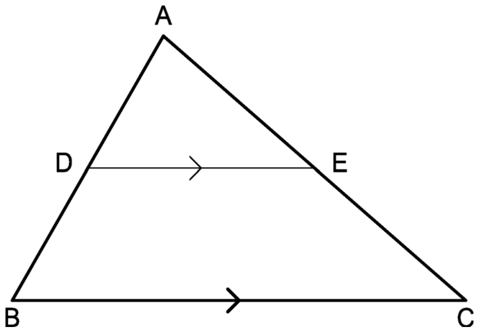
$$\frac{\text{Area } \triangle \text{PQR}}{\text{Area } \triangle \text{QRS}} = \frac{\frac{1}{2} \times 3 \times h}{\frac{1}{2} \times 4 \times h} = \frac{3}{4}$$

(b)



Area $\triangle ABC$ = Area $\triangle ABD$ since they lie between the same parallel lines ($DC \parallel AB$) and thus they have the same heights, and they share the base AB.

THE MID-POINT THEOREM

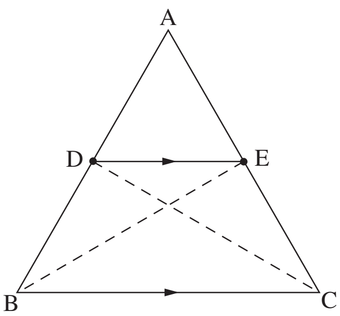


The theorem tells us if AD = DB

and $DE \parallel EC$, then AE = EC and $DE = \frac{1}{2}BC$.

THE PROPORTIONAL THEOREM

If we are given a $\triangle ABC$, with line $DE \parallel BC$ with D on AB and E on AC, then we will show that $\frac{AD}{DB} = \frac{AE}{EC}$



Join E and B and also D and C.

Now $\frac{\text{Area } \triangle ADE}{\text{Area } \triangle DEB} = \frac{\frac{1}{2}AD \times h}{\frac{1}{2}DB \times h} = \frac{AD}{DB}$

and $\frac{\text{Area } \triangle ADE}{\text{Area } \triangle DEC} = \frac{\frac{1}{2}AE \times h}{\frac{1}{2}EC \times h} = \frac{AE}{EC}$

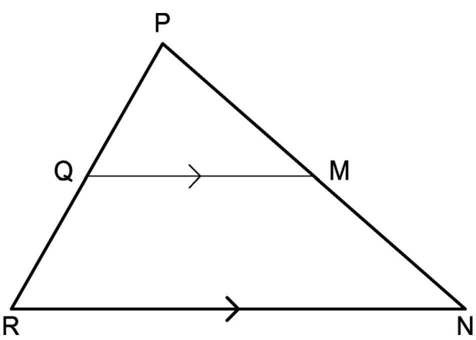
But area $\triangle DEB$ = Area $\triangle DEC$ (same height; same base)

$\therefore \frac{\text{Area } \triangle ADE}{\text{Area } \triangle DEB} = \frac{\text{Area } \triangle ADE}{\text{Area } \triangle DEC}$

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$

So to summarise:

If $OM \parallel RN$, then we can say that:



(1) $\frac{PQ}{QR} = \frac{PM}{MN}$

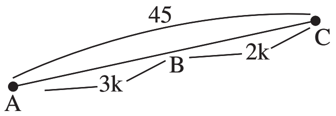
(2) $\frac{PQ}{PR} = \frac{PM}{PN}$

LET'S LOOK AT RATIO

If AB:BC = 3: 2 and AC = 45 cm, find AB and BC

Draw

Use letters (variables)



Now $5k = 45$

$k = 9$

AB = 27 cm = $3k$

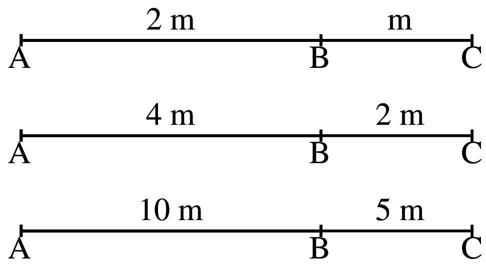
BC = 18 cm = $2k$

Ratios can be "changed"

If AB:BC = 2:1

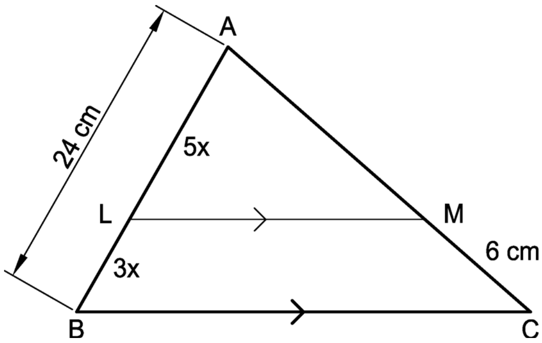
Very important

When you are given a ratio always use letters.



Examples

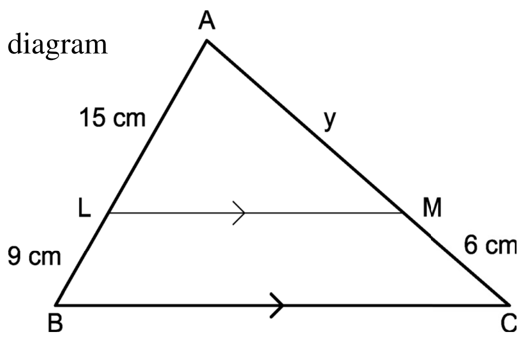
1. If AB = 24 cm and MC = 6 cm and



AL:LB = 5:3 find AL and AC

Solution

Put the information we are given into the diagram



From the diagram $8x = 24$

$x = 3$

AL = 15 cm and LB = 9 cm

Now use the theorem.

LM || BC

$\frac{\text{AL}}{\text{LB}} = \frac{\text{AM}}{\text{MC}}$ (prop. int. theorem)

$\frac{15}{9} = \frac{y}{6}$

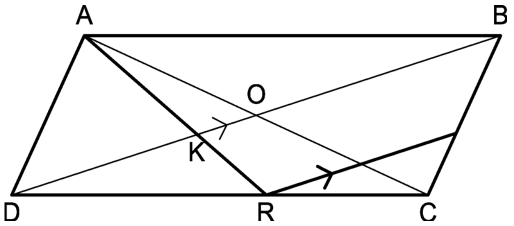
$y = 10$ cm and AC = 16 cm

Activity 1

FORMATIVE ASSESSMENT

SUMMATIVE ASSESSMENT

2. ABCD is a parallelogram.



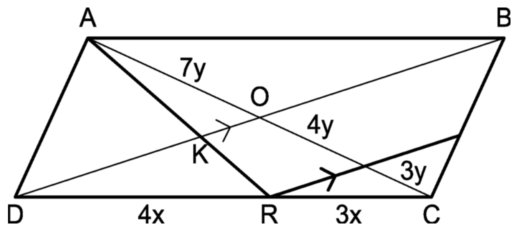
DR:RC = 4; 3

If AC = 42 cm and DB || RP

i) find the lengths of OP and PC

ii) find AK:KR

Solution



DR = $4x$ and RC = $3x$

Use the theorem

In $\triangle$ODC OD || RP

$\frac{DR}{RC} = \frac{OP}{PC} = \frac{4}{3}$ (Prop. int. theorem)

OP = 4y and PC = $3y$

But AO = OC = $7y$ diagonals bisects

i) $7y = 21$ $y = 3$

OP = 12 cm

PC = 9 cm

ii) In $\triangle$ARP

KO || RP

$\frac{AO}{OP} = \frac{AK}{KR}$ (prop. int. theorem)

$= \frac{7}{4}$

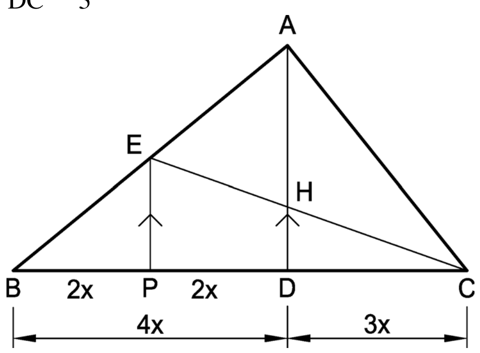
Activities 2-10

SUMMATIVE ASSESSMENT

INDIVIDUAL

3. In $\triangle$ABC, E is the mid-point of AB.

$\frac{BD}{DC} = \frac{4}{3}$ AD and CE intersect at H.



EP || AD

Calculate: i) $\frac{CH}{CE}$

ii) $\frac{\text{Area } \triangle CHD}{\text{Area } \triangle CEB}$

Solution

i) BD = $4x$ and DC = $3x$

In $\triangle$ABC EP $\parallel$ AD

$\frac{BE}{EA} = \frac{BP}{PD} = \frac{1}{1}$ (Prop. int. theorem)

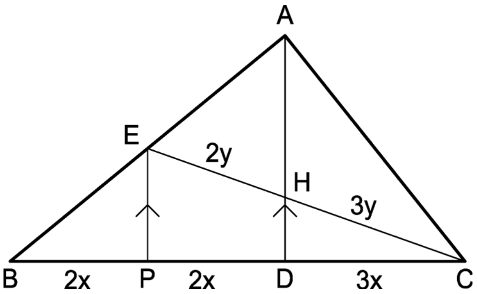
BP = PD

= $2x$

In $\triangle$EPC PE $\parallel$ DH

(Prop. int. theorem) $\frac{CH}{CE} = \frac{CD}{CP} = \frac{3}{5}$

ii) Let's draw the diagram again and use letters for sides.

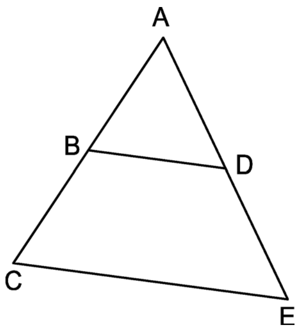


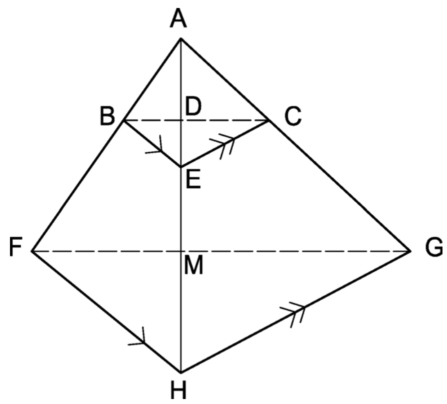
Let CH = $3y$ and HE = $2y$

$$\frac{\text{Area } \triangle CHD}{\text{Area } \triangle CEB} = \frac{\frac{1}{2} \times 3y \times 3 \times \sin C}{\frac{1}{2} \times 5y \times 7 \times \sin C} = \frac{9}{35}$$

Converse

If $\frac{AB}{BC} = \frac{AD}{DE}$ then BD $\parallel$ CE





Example

i) Prove BC $\parallel$ FG

ii) Prove $\frac{AE}{EH} = \frac{AD}{DM}$

Solution

i) $\frac{AB}{BF} = \frac{AE}{EH}$ BE $\parallel$ FH

$\frac{AE}{EH} = \frac{AC}{CG}$ EC $\parallel$ HG

$\frac{AB}{BF} = \frac{AC}{CG}$

BC $\parallel$ FG

ii) $\frac{AE}{EH} = \frac{AC}{CG}$ EC || HG

$\frac{AD}{DM} = \frac{AC}{CG}$ EC || MG

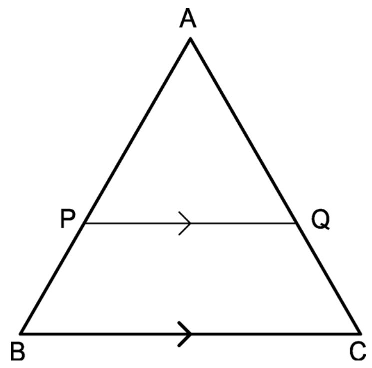
$\frac{AE}{EH} = \frac{AD}{DM}$

Activity

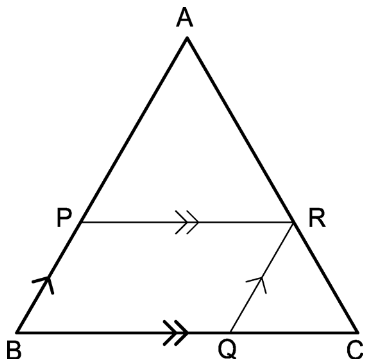
FORMATIVE ASSESSMENT

PAIRS

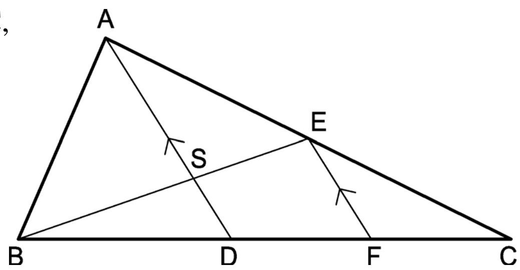
1. AP:AB = 5:9. If QC = 24 cm, find the length of AQ.



2. Prove $\frac{AP}{PB} = \frac{BQ}{QC}$



3. AD and BE are medians of $\triangle ABC$,
   i) Find $\frac{BS}{SE}$
   ii) Find $\frac{BF}{FC}$



4. PQRS is a parallelogram

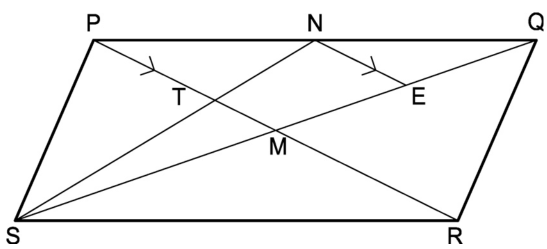
   PN:NQ = 3; 5

   Find

   i) $\frac{KQ}{SQ}$

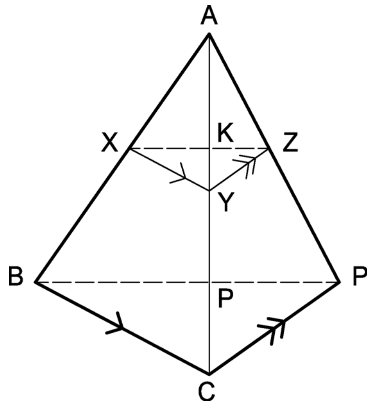
   ii) $\frac{SM}{MK}$

   iii) If ST = 32 cm, find the length of SN.



5. Prove that: (a) XZ || BP

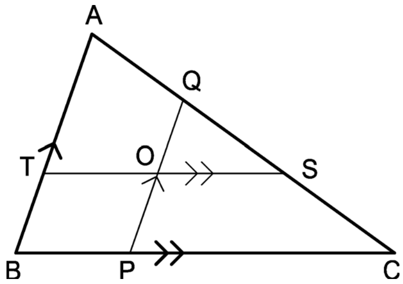
(b) $\frac{AY}{YC} = \frac{AK}{KM}$



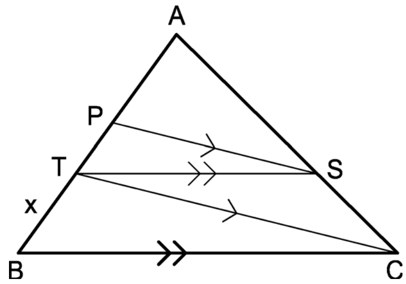
6. In $\triangle ABC$, AC = 160 mm $\frac{AT}{TB} = \frac{5}{3}$ and $\frac{P}{PC} = \frac{1}{3}$

a) Find the length of QS

b) Prove O is the mid-point pf PQ.



7. AP = 50 mm, PT = 30 mm. Find x.



GEOMETRY (2)

Similar triangles and other figures

Lesson 32

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and movement.
Assessment Standard
a) Investigate necessary and sufficient conditions for polygons to be similar.
b) Prove and use (accepting results proved in earlier grades):
- That a line drawn parallel to one side of a triangle divides the other two sides proportionally, (the mid–point theorem is a special case of this theorem)
- That equiangular triangles are similar.
- That triangles with sides in proportion are similar.
- The Pythagorean theorem by similar triangles.

Overview

In this lesson you will:

- Establish a definition for similarity.
- Investigate similar figures and decide whether they are similar or not.
- Show that all triangles that are equiangular are similar.
- Use the ratios of proportional sides to solve problems.

Lesson

LC DVD

Investigation 1

1. Construct $\triangle ABC$ with AB = 10 cm, BC = 7 cm and AC = 6 cm and $\triangle PQR$ with PQ = 5 cm, QR = 3,5 cm and PR = 3 cm.
   a) Measure all the angles and what do you notice?
   b) Which triangle is enlarged?
   c) What is $\frac{AB}{PQ}$?
   d) What is $\frac{BC}{QR}$?
   e) What is $\frac{AC}{PR}$?
   f) What do you notice?
2. Draw any two squares ABCD and PQRS. Are all the angles equal?
   Is $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{DC}{SR} = \frac{AD}{PS}$. Do you think they are similar?
3. Draw any two rectangles that are not squares ABCD and PQRS. The angles are all equal.
   Is $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{DC}{SR} = \frac{AD}{PS}$. Do you think they are similar?
4. Draw any two rhombi ABCD and PQRS
   Is $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{DC}{SR} = \frac{AD}{PS}$. Are they equiangular? Do you think they are similar?

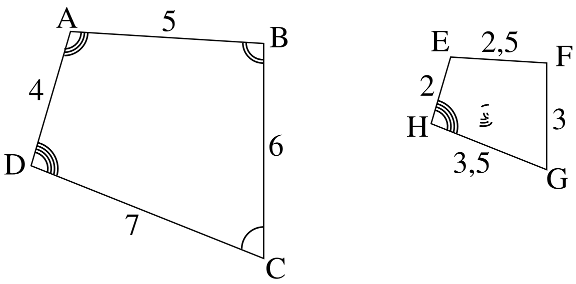
LC Maths G11 WB.indb 155 2008/08/09 08:26:00 AM

Similarity of polygons

Two polygons with the same shape are called **similar polygons.** Two polygons are similar, if and only if the following two facts *are both true*:

- Corresponding angles are equal.
- The ratios of pairs of corresponding sides must all be equal.

Example 1: Working with similar quadrilaterals



It is clear to see that the corresponding angles are all equal.

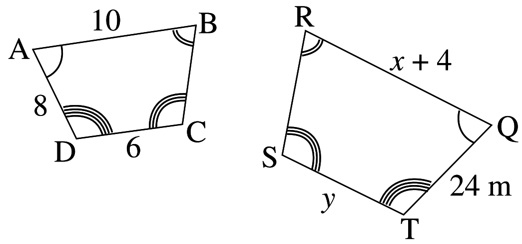
That is $\hat{A} = \hat{E}$, $\hat{B} = \hat{F}$, $\hat{C} = \hat{G}$ and $\hat{D} = \hat{H}$.

If we check the proportionality of the sides:

$\frac{AB}{EF} = \frac{5}{2,5} = \frac{2}{1}$; $\frac{BC}{FG} = \frac{6}{3} = \frac{2}{1}$; $\frac{DC}{HG} = \frac{7}{3,5} = \frac{2}{1}$; $\frac{AD}{EH} = \frac{4}{2} = \frac{2}{1}$.

Example 2

If the two quadrilaterals alongside are given similar, find the values of x and y. Also find the scale factor of polygon $ABCD$ to polygon $QRST$.



Solution

$\frac{AB}{RQ} = \frac{AD}{QT} \rightarrow \frac{10}{x + 4} = \frac{8}{24} \rightarrow 8(x + 4) =$ 240

$\therefore x + 4 = 30$

$\therefore x = 26$

$\frac{DC}{ST} = \frac{AD}{QT} \rightarrow \frac{6}{y} = \frac{8}{24} \rightarrow 8y = 144 \rightarrow y = 18$

Scale factor: This is the ratio of the length of two corresponding sides.

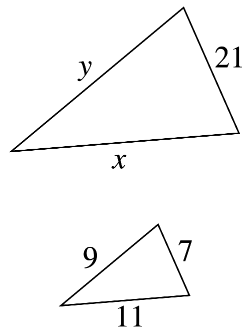
So $\frac{AD}{QT} = \frac{8}{24} = \frac{1}{3}$.

Two interesting facts about polygons that are similar:

- If two polygons are similar, their corresponding sides, altitudes, medians, diagonals, and perimeters are all in the same ratio.
- If two polygons are similar, the ratio of their **areas** is equal to the **square** of the ratio of their corresponding sides.

Example 3

Two triangles are similar. The sides of the first triangle are 7, 9, and 11. The smallest side of the second triangle is 21. Find the perimeter of the second triangle. Also find the ratio of their areas using the perimeters of the triangles.



Solution

Perimeter = $x + y + 21$

For x and y:

$\frac{11}{7} = \frac{x}{21} \rightarrow x = \frac{11 \times 21}{7} = 33$

$\frac{9}{7} = \frac{y}{21} \rightarrow y = \frac{9 \times 21}{7} = 27$

So perimeter = 33 + 27 + 21 = 81.

For the areas:

$\frac{\text{area triangle 1}}{\text{area triangle 2}} = \frac{(\text{perimeter}\Delta_1)^2}{(\text{perimeter}\Delta_2)^2} = \frac{(81)^2}{(27)^2} = \frac{3^8}{3^6} = \frac{9}{1}$

So area triangle 1 = 9 × area triangle 2

Triangles are special

If two triangles are equiangular, their corresponding sides are in proportion.

We will prove this.

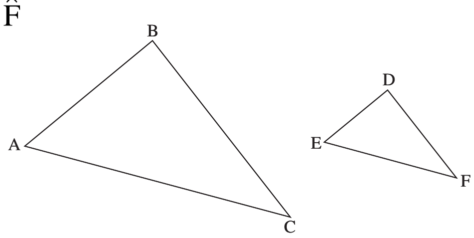
Given $\triangle$sABC and DEF $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$

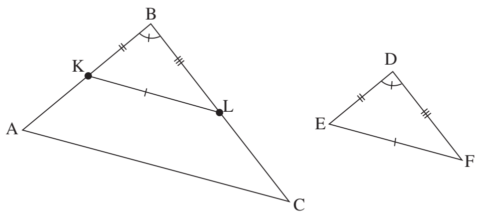
RTP: $\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC}$

Proof: On AB mark off K such that AK = DE

On AC mark off L such that AL = DF

Join KL.





In $\triangle$KAL and $\triangle$EDF:

AK = DE	Construction
AL = DF	Construction
$\hat{A} = \hat{D}$	Given
$\therefore \triangle KAL \equiv \triangle EDF$ SAS	

So $\hat{K}_1 = \hat{E}$

But $\hat{E} = \hat{B}$	Given
Thus $\hat{K}_1 = \hat{E}$	
$\therefore KL \parallel BC$	Corresponding angles equal
So now: $\frac{AK}{AB} = \frac{AL}{AC}$	Sides in proportion

But AK = DE and AL = DF Construction

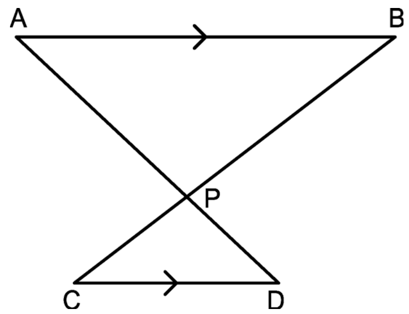
$\therefore \frac{DE}{AB} = \frac{DF}{AC}$

In a similar way, by marking off on BC, BM = EF, we can show that

$\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC}$.

So we can conclude that if two triangles are equiangular, their corresponding sides are in proportion.

Look at this picture



In $\triangle$s ABP and PCD

1. $\hat{A} = \hat{D}$ (alternate angles AB $\parallel$ CD)
2. $\hat{B} = \hat{C}$ (alternate angles AB $\parallel$ CD)
3. $\hat{P} = \hat{P}$ (angles in a triangle)

$\therefore \triangle ABP \parallel\!\mid \triangle DCP$

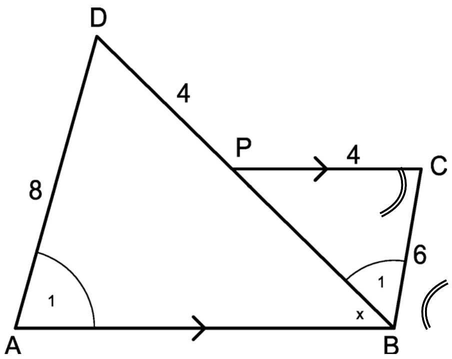
$\therefore \frac{AB}{DC} = \frac{AP}{DP} = \frac{BP}{CP}$

Example

$\hat{B}1 = \hat{A}$ PC = 4 cm BC = 6 cm

DP = 4 cm and AD = 8 cm

Find PB



Solution

Let's put the information in the diagram.

In $\triangle$s ADB and PBC

$\hat{A} = \hat{B}1$	Given
$\hat{B}_2 = \hat{C}$	Alternate angles
$\therefore \hat{D} = \hat{C}$	Angles in triangle
$\therefore \triangle ABD \parallel\!\mid \triangle BCP$	Equiangular
$\therefore \frac{AB}{BC} = \frac{BD}{CP} = \frac{AD}{BP}$	Sides in prop
$\therefore \frac{BD}{CP} = \frac{AD}{BP}$	
$\frac{PB + 4}{4} = \frac{8}{PB}$	Cross multiply

PB(PB + 4) = 32

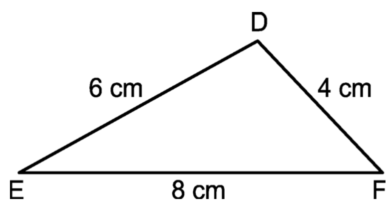
$PB^2 + 4PB - 32 = 0$

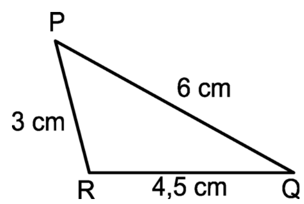
(PB + 8)(PB – 4) = 0

PB = 4 units

Converse

If the corresponding sides of two triangles are in proportion, they are similar (in other words equiangular)





Example

$\frac{DF}{PR} = \frac{4}{3}$

$\frac{DE}{RQ} = \frac{6}{4,5} = \frac{4}{3}$

$\frac{EF}{PQ} = \frac{8}{6} = \frac{4}{3}$

$\therefore \hat{E} = \hat{Q}$

$\hat{D} = \hat{R}$

So $\triangle EFD \,|||\, \triangle QPR$

Activities 2–8

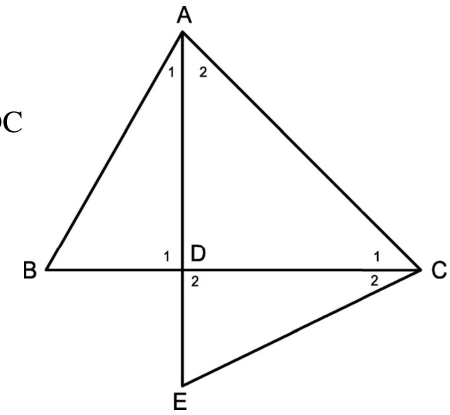
FORMATIVE ASSESSMENT

PAIRS

1. AE bisects $B\hat{A}C$ and $\hat{B} = \hat{E}$

   Prove a) $\triangle ABD \,|||\, \triangle AEC$

   b) $AB.AC = AD^2 + BD.DC$

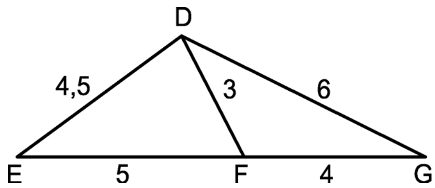


SUMMATIVE ASSESSMENT

INDIVIDUAL

2. a) Prove $\triangle GDF \,|||\, \triangle GED$

   b) Which angles are equal in size?



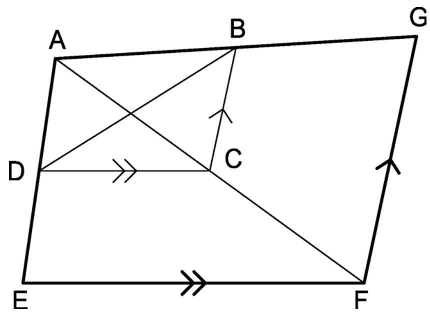
3. ABCD is a parallelogram

   EF $\parallel$ DC and FG $\parallel$ CB

   a) Calculate AC if AD = 12 cm
   DE = 3 cm and CF = 5 cm.

   b) Calculate EF if DC = 14 cm

   c) Calculate BG



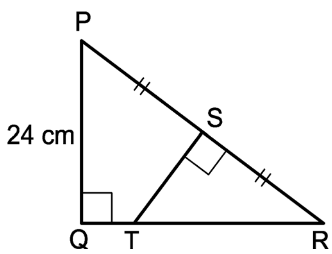
4. PR = 40 CM and PS = SR

   Calculate

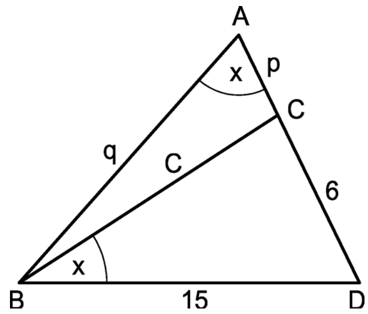
   a) QR

   b) ST

   c) QT

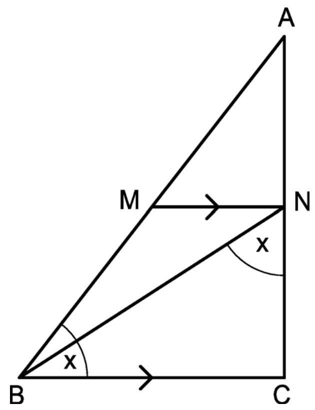


5. Calculate the lengths of AB and AC

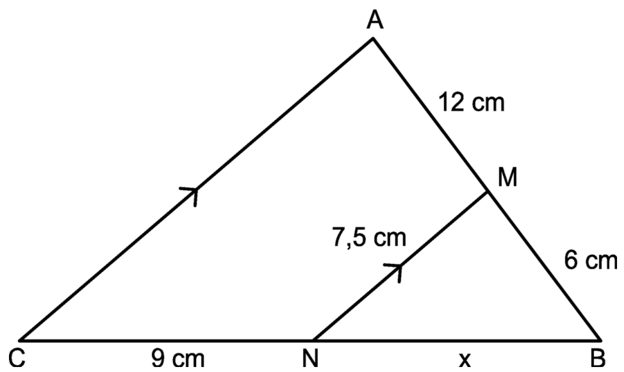


6. a) Prove that $BC^2 = AC.NC$

   b) Prove that $AM.NC = BN.MN$

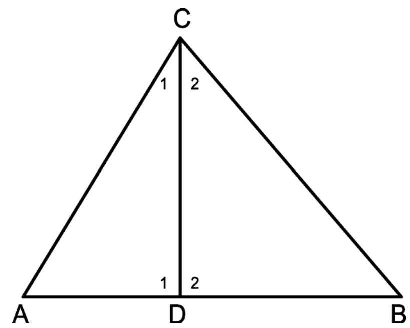


7. Prove that $\triangle ABC$ is a right–angled triangle.



8. AD.BC = CD.AC and
   AC.BD = CD.BC

   Prove $AB^2 = AC^2 + BC^2$



GEOMETRY (3)

Application of similar triangles

Lesson 33

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, Shape and Measurement
Assessment Standard
- Prove and use the fact that
- Equiangular triangles are similar
- That triangles with sides in proportion are similar
- The Pythagorean theorem by similar triangles

Overview

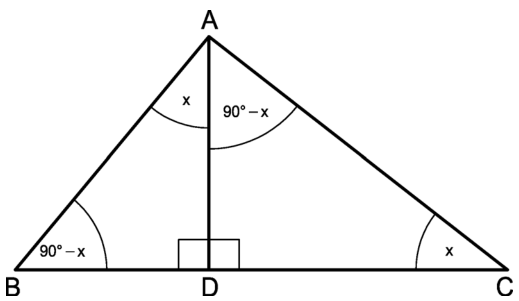
In this lesson you will:

- Prove that in a right angled triangle, the line drawn from the right angle perpendicular to the hypotenuse divides the original triangle into three triangles that are all similar to each other.
- Make conclusions from this theorem and use these conclusions to solve geometrical problems.
- Prove the theorem of Pythagoras.
- Use the theorem of Pythagoras to solve problems.

Lesson

DVD

Look at this diagram



In $\triangle$'s ABC, ABD and ACD

$\hat{C} = \hat{A} = \hat{C}$

$\hat{A} = \hat{D} = \hat{D}$

$\hat{B} = \hat{B} = \hat{A}$

$\therefore\ \triangle CAB \,|||\, \triangle ADB \,|||\, \triangle CDA$

From above

$\frac{AB}{DB} = \frac{CD}{AB}$

$\therefore\ AB^2 = CD \cdot DB$

Above we proved that

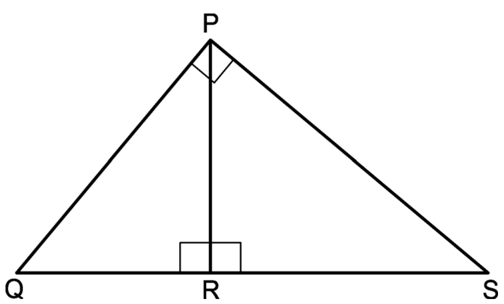
$\therefore \triangle CAB \,|||\, \triangle ADB \,|||\, \triangle CDA$

$\therefore \triangle CAB \,|||\, \triangle CDA$ $\qquad$ $\triangle ADB \,|||\, \triangle CDA$

$\frac{CA}{CD} = \frac{CB}{CA}$ and $\frac{AD}{CD} = \frac{DB}{AD}$

$AC^2 = CB \cdot CD$ so

$AD^2 = DB \cdot CD$



How do we use this theorem

$PS^2 = SR \times SQ$

$PR^2 = RQ \times RS$

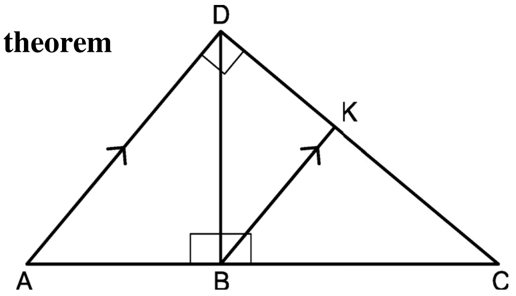
But by Pythagoras we can also say

$PQ^2 = PR^2 + QR^2$

Or

$PQ^2 = QS^2 - PS^2$

Let's do an example of how we use this theorem

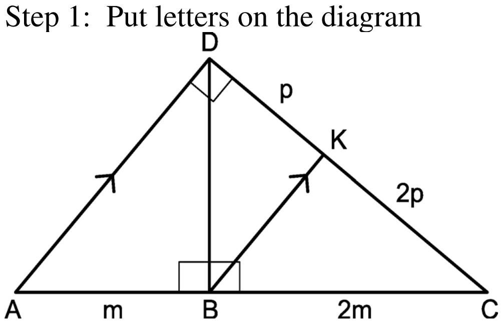


In $\triangle ADC$ (D) = 90° $DB \perp AC$

$AD \parallel BK$ and $2DK = KC$

Prove $2DA = \sqrt{3BC}$

Step 1: Put letters on the diagram



$AD \parallel BK$

$\frac{DK}{KC} = \frac{AB}{BC}$ Prop. Int. theorem

$= \frac{m}{2\ m}$

$DA^2 = AB.AC$ Perp from right angle to hyp

$= m \times 3\ m$

$= 3\ m^2$

$\therefore\ DA = \sqrt{3\ m}$ We want 2DA so

$2DA = 2\sqrt{3\ m}$

$= \sqrt{3} \cdot 2\ m$

$= \sqrt{3BC}$

From the theorem we can prove the theorem of Pythagoras

Prove: $BC^2 = AB^2 + AC^2$

Construct $AD \perp BC$

$AB^2 = BD \times BC$ Perp. from right < to hyp.

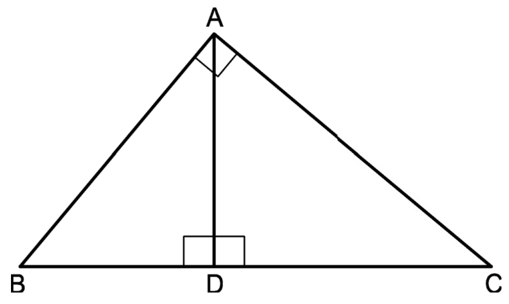
$AC^2 = AC \times CB$ Perp. form right < to hyp.

$AB^2 + AC^2 = BD \cdot BC + CD \cdot CB$

$AB^2 + AC^2 = BC(BD + CD)$

$= BC \times BC = BC^2$

$AB^2 + AC^2 + BC^2$

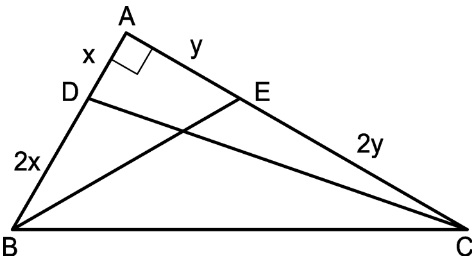


Let's do an example by using Pythagoras in geometry

$\hat{A} = 90°$ $2AD = DB$ and $2AE = EC$

Prove $9(DC^2 + BE^2) = 10\ BC^2$

Put letters on the diagram



LHS	RHS
$DC^2 = x^2 + 9y^2$	$BC^2 = 9x^2 + 9y^2$
$BE^2 = 9x^2 + y^2$	$10BC^2 = 90x^2 + 90y^2$
$DC^2 + BE^2 = 10x^2 + 10y^2$	$\therefore$ LHS = RHS
$9(DC^2 + BE^2) = 90x^2 + 90y^2$	

Activity 1

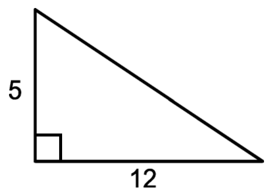
SUMMATIVE ASSESSMENT

INDIVIDUAL

DVD

Where have we used Pythagoras? In trigonometry.

1. If $3 \tan \alpha = 4$ and $\alpha \in [\ 180° ; 360°\ ]$ without a calculator find $\sin \alpha$
2. Triangles in trigonometry

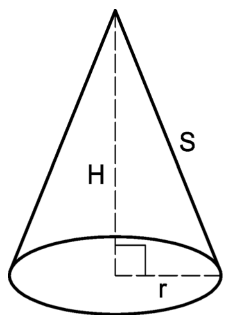


3. Analytical Geometry

   $d = R(x^2 - x^1)^2 + (y^2 - y^1)^2$ is derived from Pythagoras
4. Surface area and volume

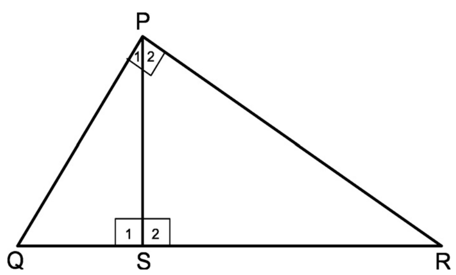
   To find the slant height S

   $S^2 = H^2 + r^2$



Activity 2

1. Prove: $\dfrac{PQ^2}{PR^2} = \dfrac{QS}{RS}$



2. Complete the following statements.

i) QR^2 = ________ × ________

ii) QR^2 = ________ + ________

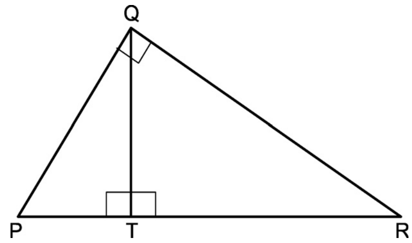
iii) QR^2 = ________ - _________

iv) QP^2 = ________ × _________

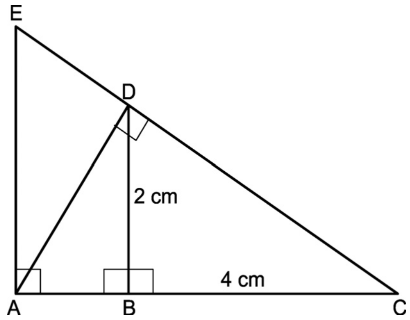
v) QP^2 = ________ + _________

vi) QP^2 = ________ - _________

vii) QT^2 = ________ × _________

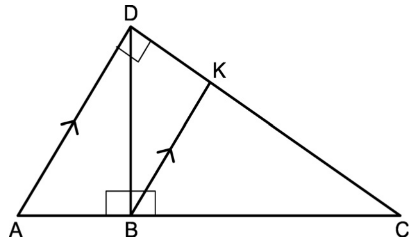


3. Find the length of EA



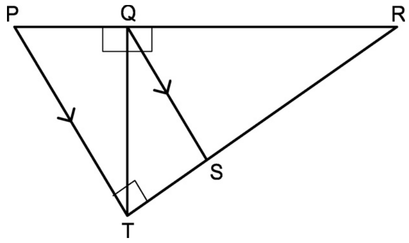
4. 3DK = KC

Prove 2AD = AC



5. 4TS = SR

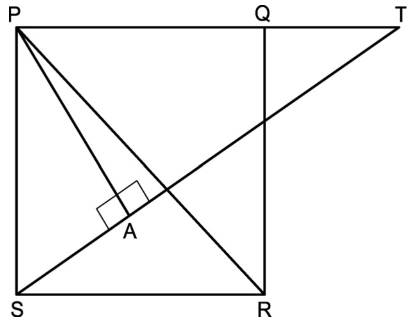
Prove 4TP = $\sqrt{5}$ QR



6. PQRS is a square. PA ⊥ ST

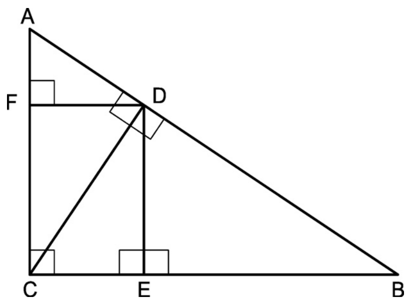
i) Prove that PR = $\sqrt{2}$ PS

ii) If PT = PR,
prove that TA = 2AS



7. Prove: i) $\dfrac{BD}{AD} = \dfrac{BC^2}{AC}$

ii) $\dfrac{BE}{AF} = \dfrac{BC^3}{AC^3}$



Investigation

Show that is Pythagoras's theorem the squares on the sides of the right-angled triangles can be replaced by circles.

Investigate other possibilities.

TRANSLATION OF GRAPHS (1)

The parabola and hyperbola

Lesson 34

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment Standard
Generate as many graphs as necessary, initially by means of point plotting, supported by available technology, to make and test conjectures about the effect of the parameters k, p, a and q for the functions including:

$y = \sin kx$	$y = \cos kx$
$y = \tan kx$	$y = \sin(x + p)$
$y = \cos(x + p)$	$y = \tan(x + p)$
$y = a(x + p)^2 + q$	
$y = abx + p + q$	$y = \frac{a}{x + p} + q$

Overview

In this lesson you will:

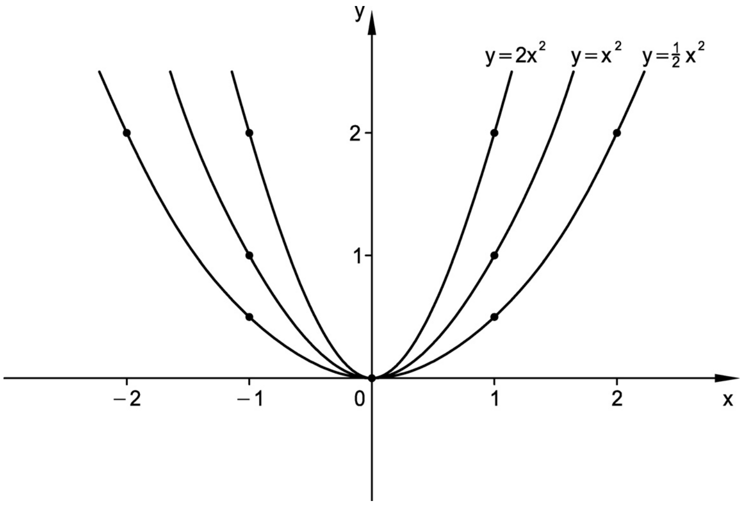
- Draw the functions you learnt in Grade 10 and exercise vertical translation and reflection across both the x- and y-axes.
- Translate graphs horizontally along the x-axis.
- Translate graphs both vertically and horizontally.

Lesson

LC DVD

Revision

Let's look at $y = x^2$; $y = \frac{1}{2}x^2$ and $y = 2x^2$ on the same set of axes.

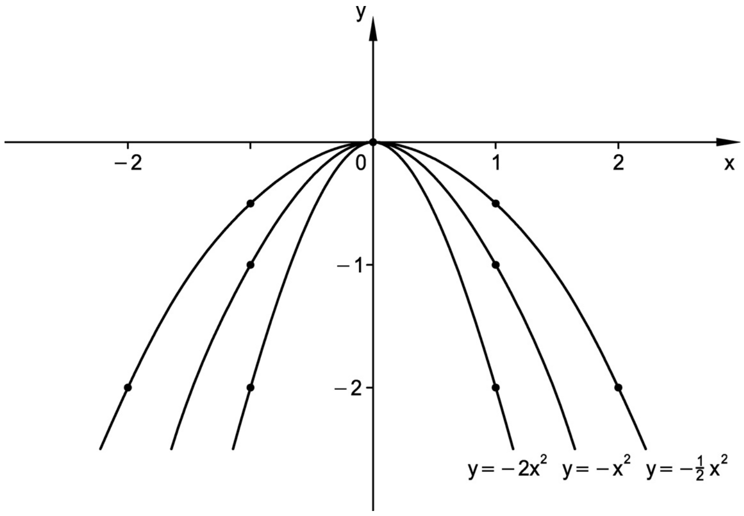


Conclusion: $y = ax^2$

The graph is stretched vertically by a factor of a.

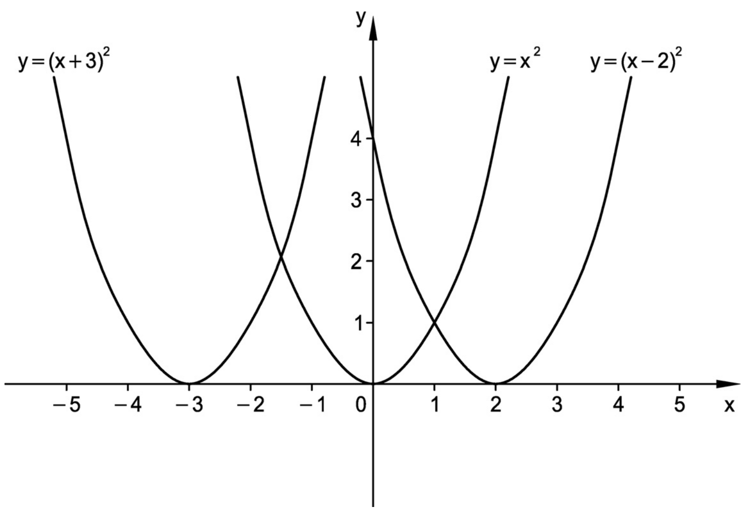
Let us look at:

$y = –x2$; $y = -\frac{1}{2}x^2$ and $y = –2x^2$

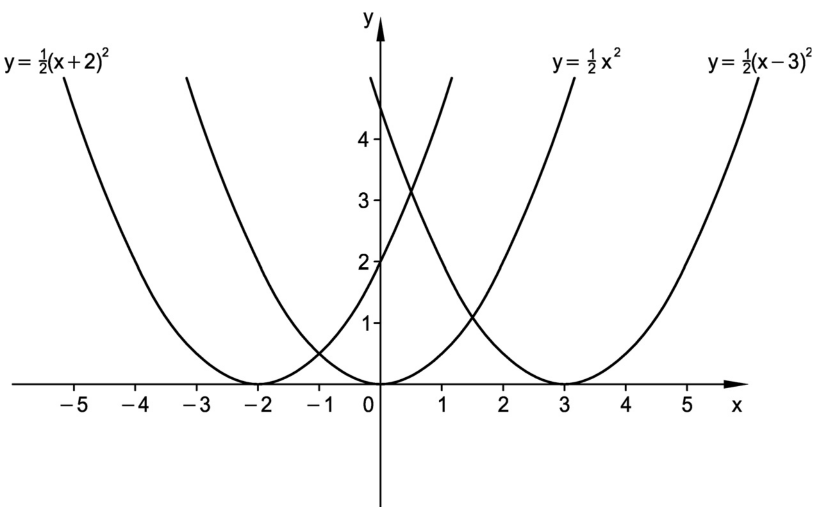


The graph is stretched vertically by a factor of a and reflected about the *x*–axis.

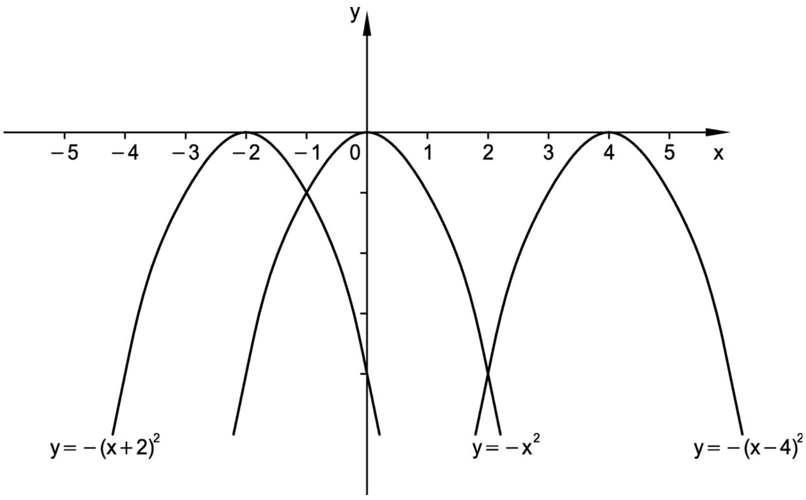
We will keep playing with the parabola but we will translate it along the *x*–axis and see what happens.



Let's translate the others



Reflect and translate

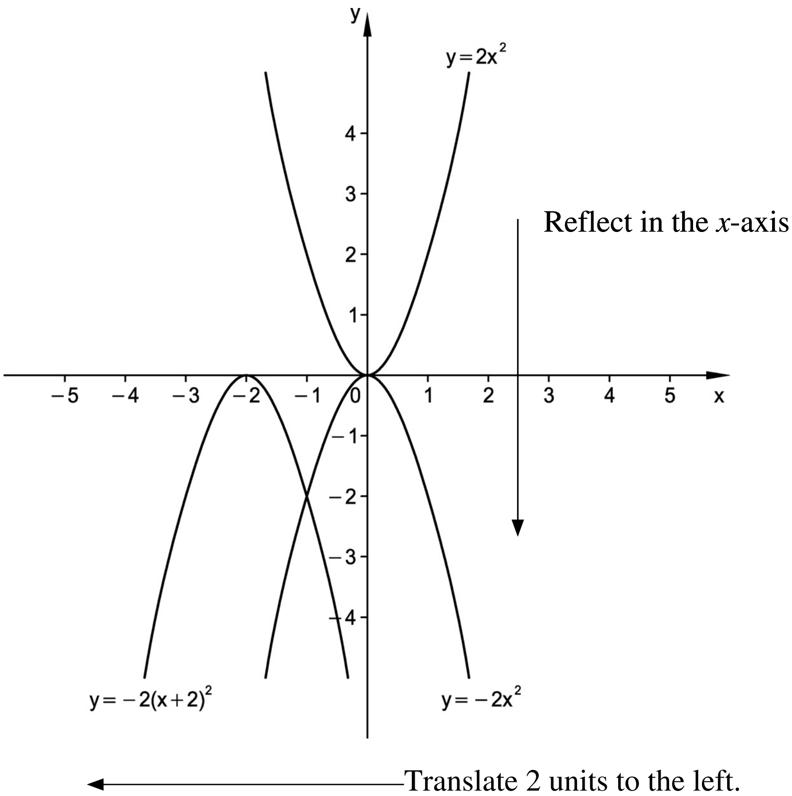


Now what will you do if you are asked to draw $y = -2(x + 2)^2$

Step 1:

Draw $y = 2x^2$ and reflect it

To be $y = -2x^2$

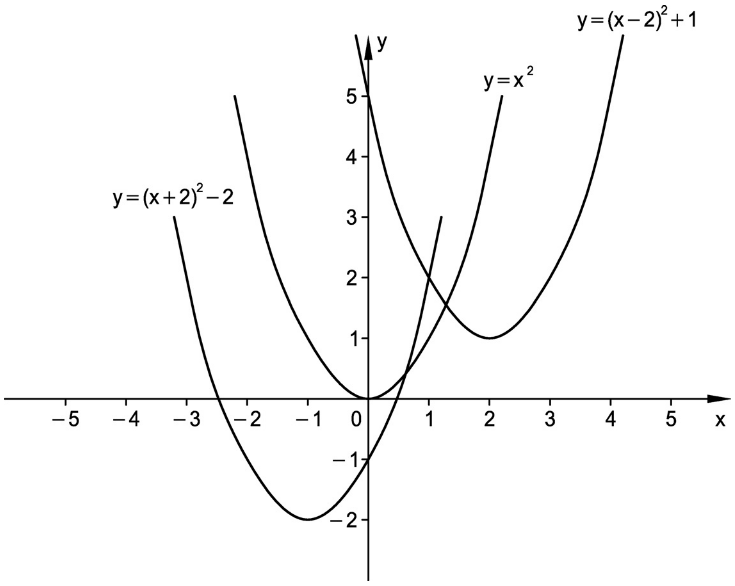


Step 2:

Translate it 2 units to the left.

Activity 1

How about translating a parabola horizontally and vertically?

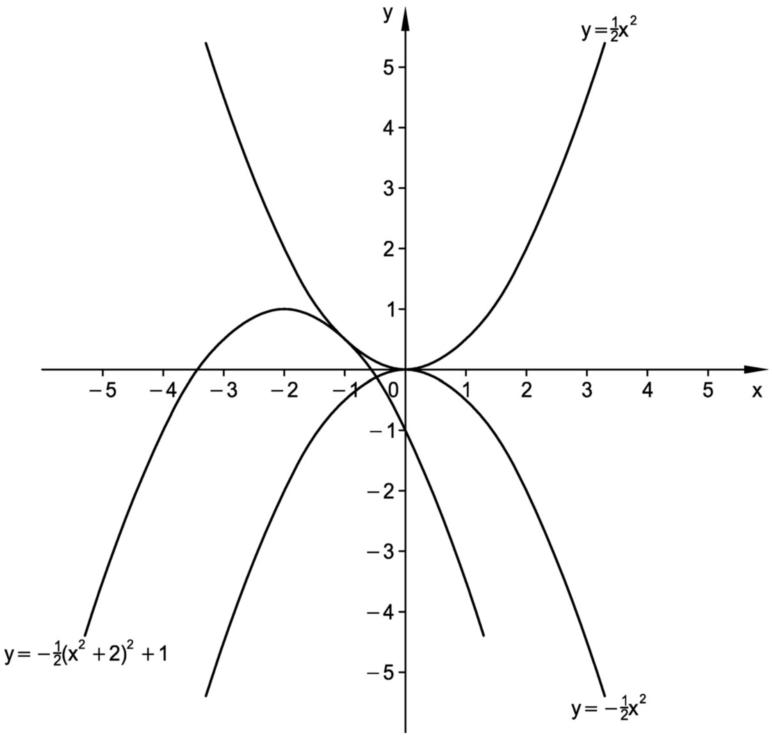


We will translate $y = x^2$ to the right 2 and up 1. What will the equation be?

We will translate $y = x^2$ to the left 1 and down 2. What will the equation be?

How will you draw $y = -\frac{1}{2}(x + 2)^2 + 1$ by translating the graph of $y = x^2$?

Draw $y = \frac{1}{2}x^2$ and reflect it across the x–axis to become $y = -\frac{1}{2}x^2$

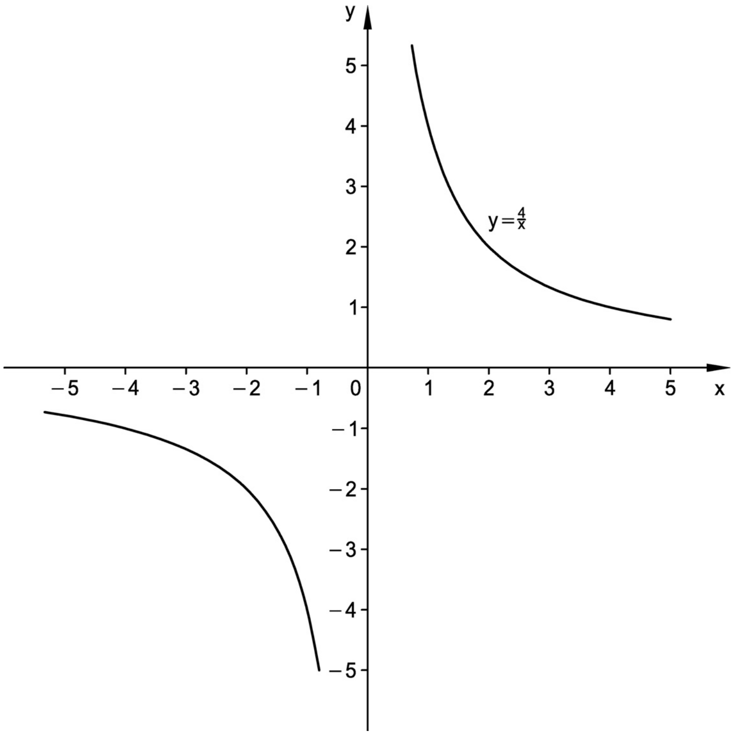


Now translate it 2 units left and one unit up.

Activity 2

How about the other graphs you learnt last year?

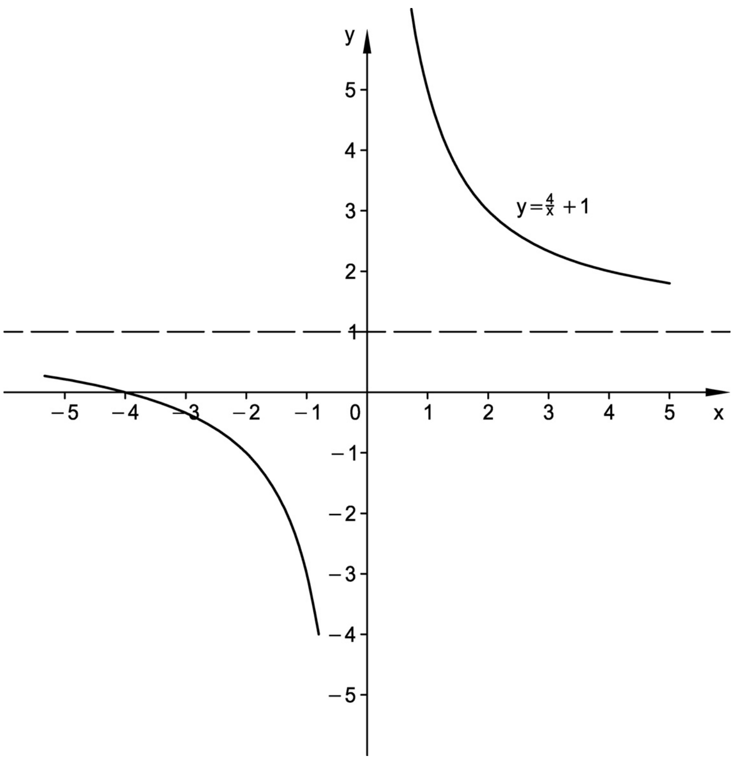
Hyperbola: $y = \frac{4}{x}$



Asymptotes at $x = 0$ and $y = 0$

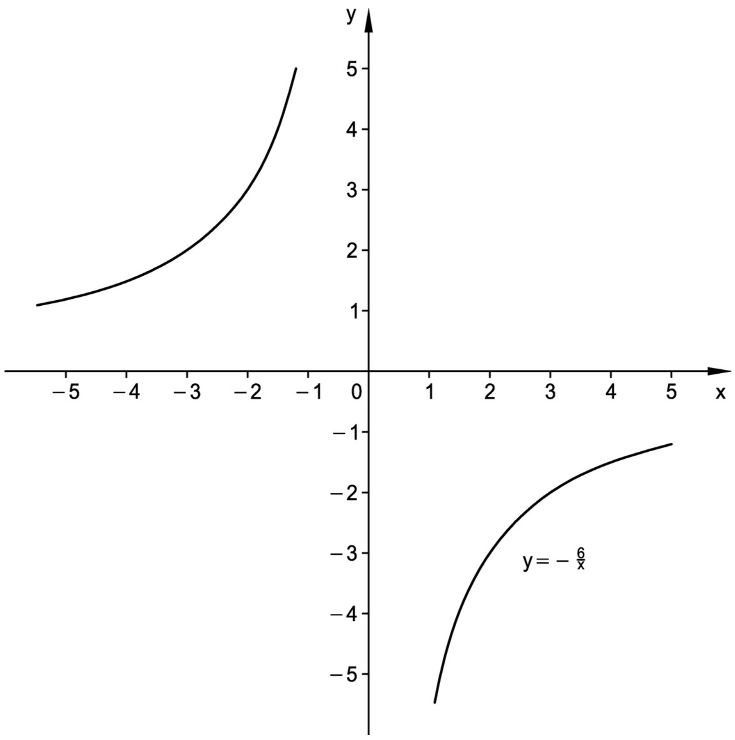
What about $y = \frac{4}{x} + 1$?

(We translate the whole graph, including the asymptote up 1).

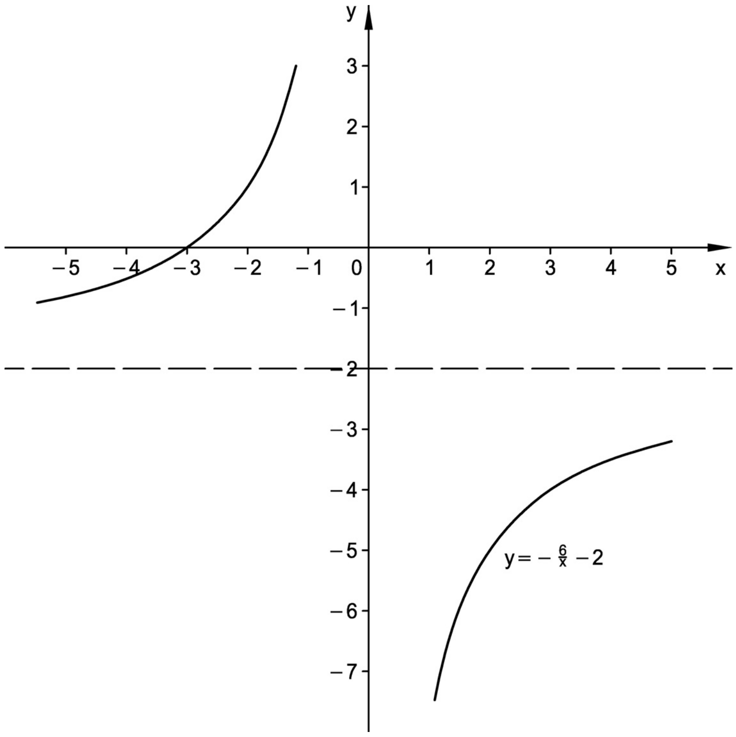


Remember this one $y = -\frac{6}{x} - 2$

First draw $y = -\frac{6}{x}$



Now translate the graph (including the asymptote) down 2.

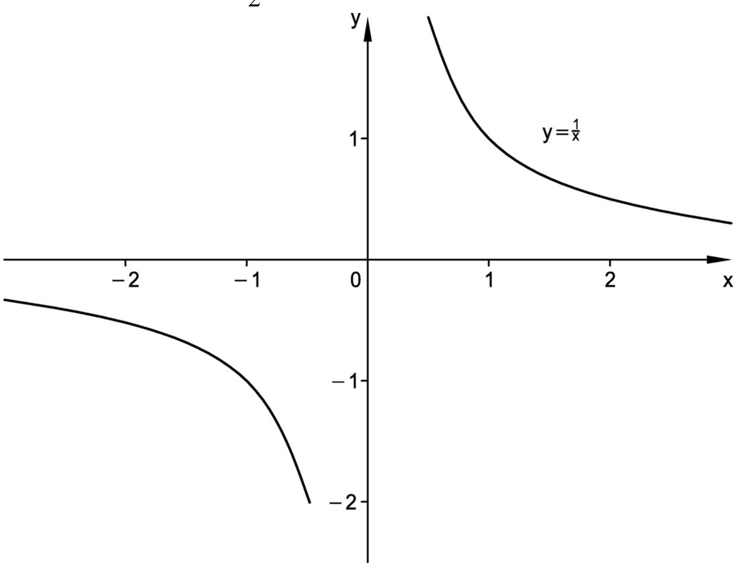


Now we will learn to translate the hyperbola horizontally.

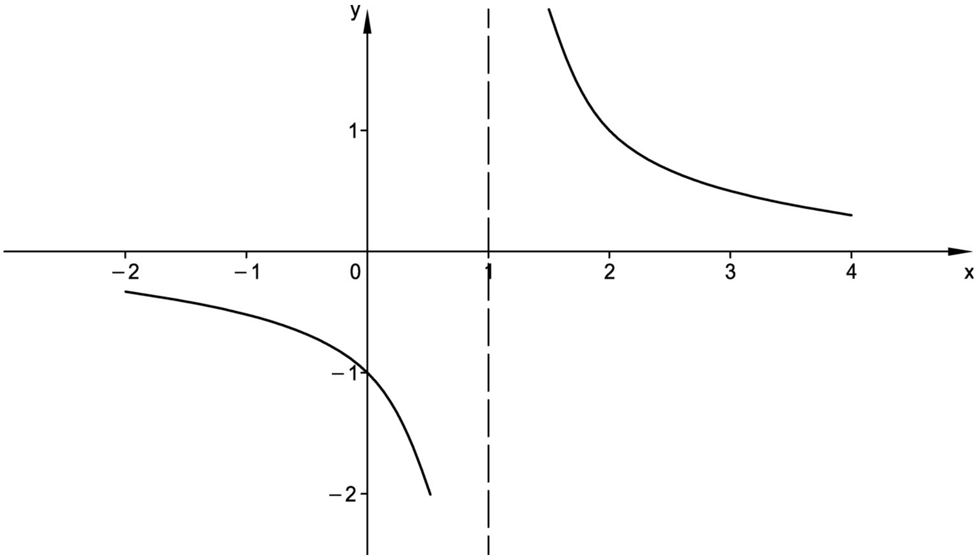
$y = \frac{1}{x-1}$

What do you think we will do? We translate $y = \frac{1}{x}$ one unit to the right.

Let's first draw $y = \frac{1}{2}$



Now translate the whole graph (including the asymptote) 1 unit to the right.



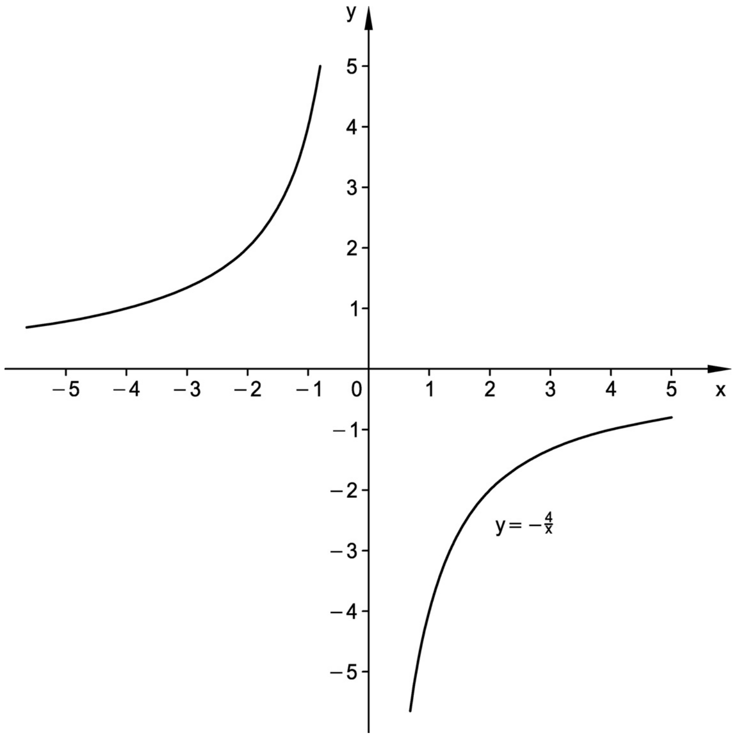
Let's do another horizontal translation.

Sketch $y = \frac{-4}{x+2}$

Translation plan:

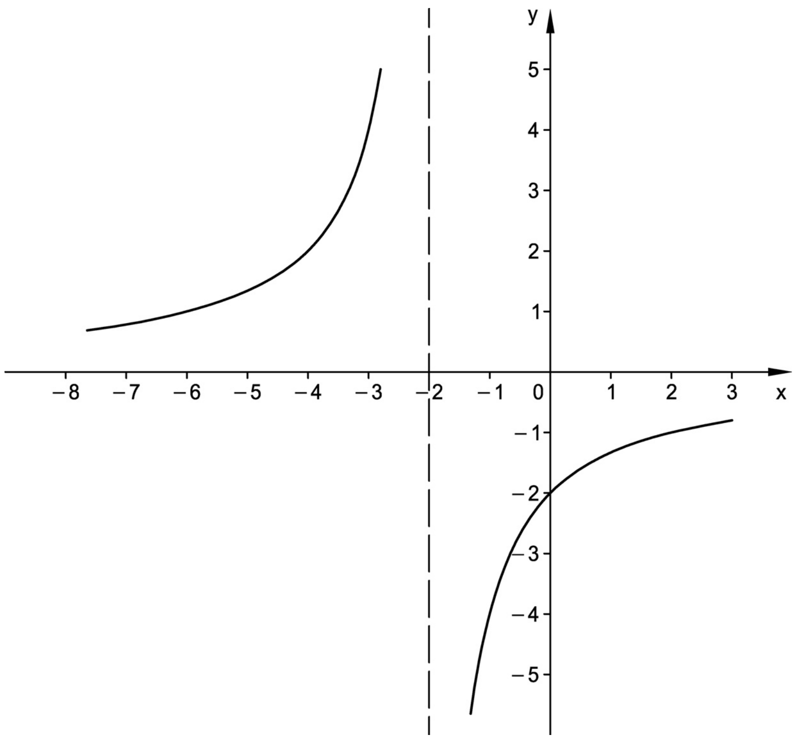
Step 1:

Reflect $y = \frac{4}{x}$ across the y–axis to become $y = -\frac{4}{x}$



Step 2:

Now translate this graph 2 units to the left (asymptote first).

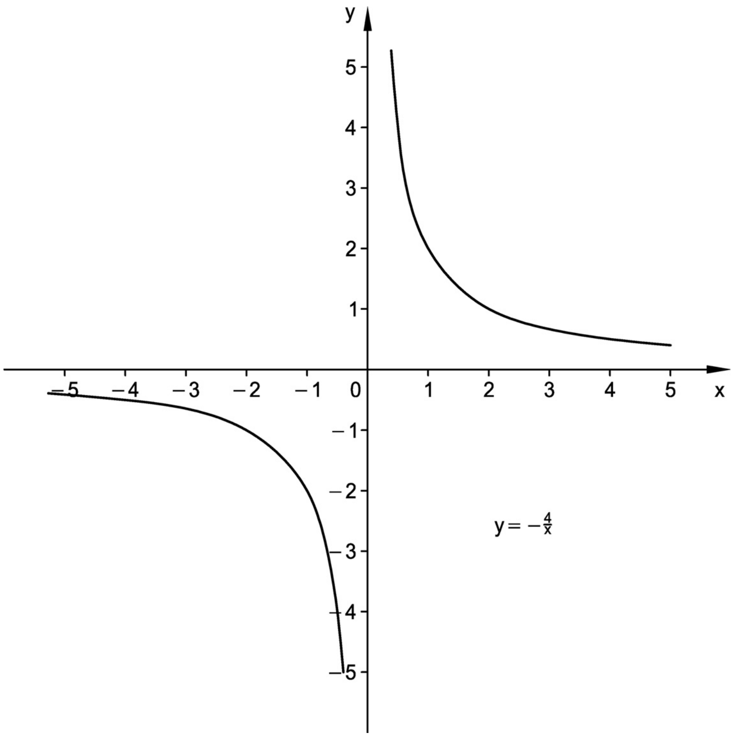


What about translating the hyperbola both vertically and horizontally?

Sketch: $y = \frac{2}{x + 1} - 2$

Translation plan:

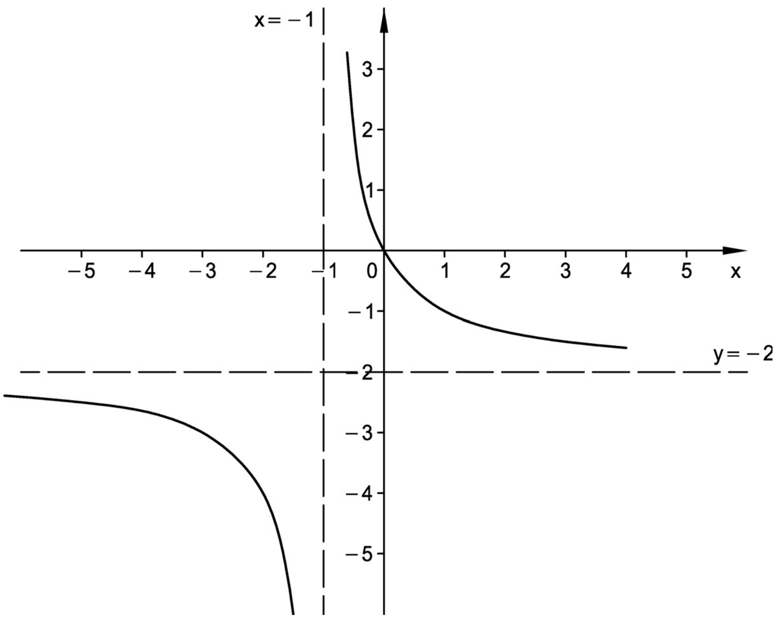
Step 1: Draw $y = \frac{2}{x}$



Step 2:

Now translate the graph 1 unit to the left and 2 units down.

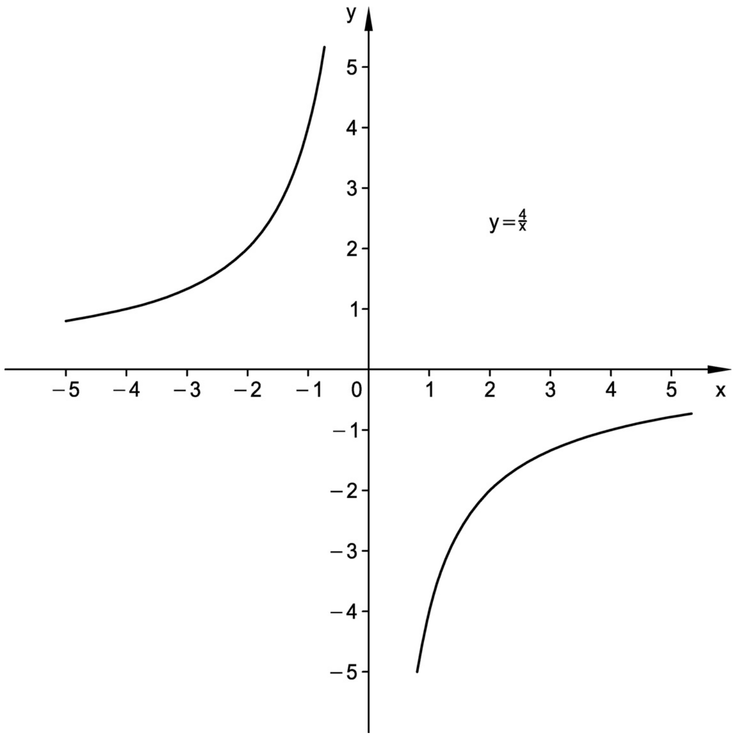
(First name the asymptotes).



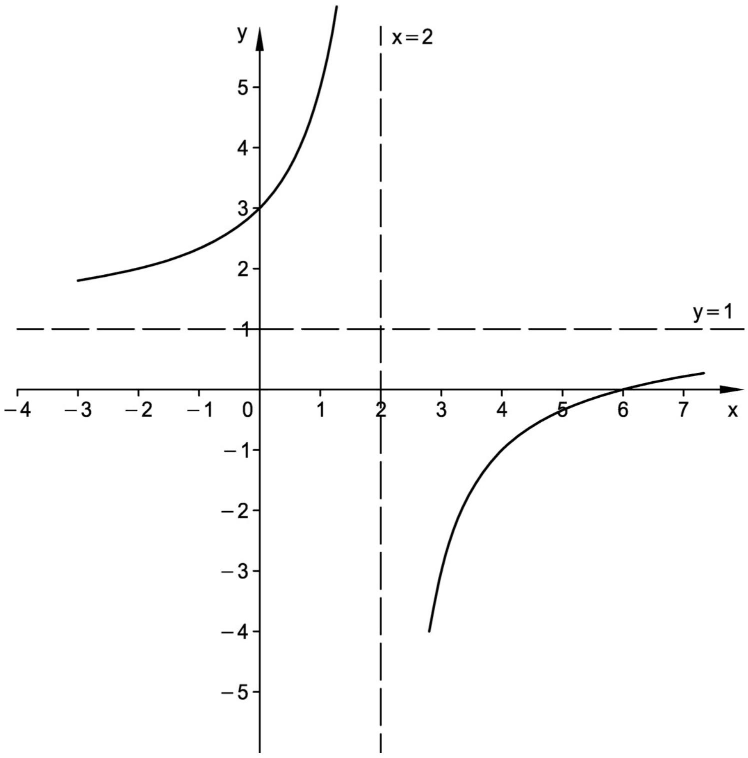
Another one: Sketch $y = \frac{-4}{x-2} + 1$

Translation plan:

Sketch $y = \frac{4}{x}$ and reflect it across the y–axis to become $y = -\frac{4}{x}$.



Now translate the graph 2 units to the right and 1 unit up (asymptotes first).



Summary

Parabola: $a(x - p)^2 + q$

Draw $y = ax^2$

Move the turning point to $(p\ ;\ q)$

Keep the shape the same as $y = ax^2$

Hyperbola: $y = \frac{k}{x - p} + q$

Draw: $y = \frac{k}{x}$

Move the asymptotes $x = p$ $y = q$

Find any x–intercepts ($y = 0$)

Find any y–intercepts ($x = 0$)

Draw the arms of the hyperbola

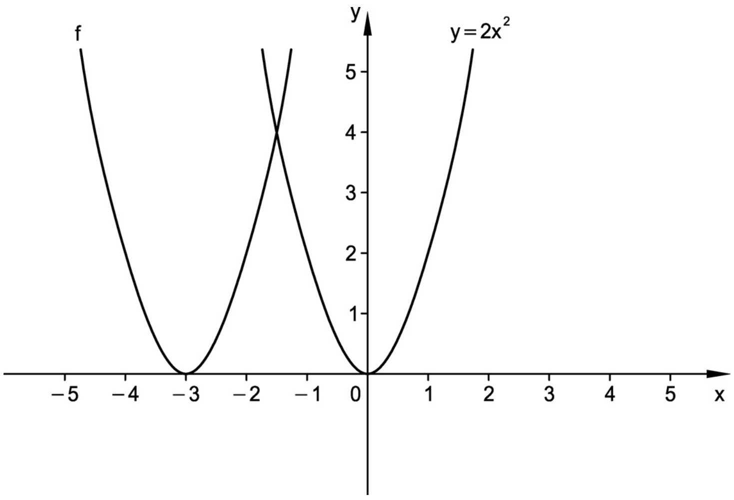
Activity 1

PAIRS

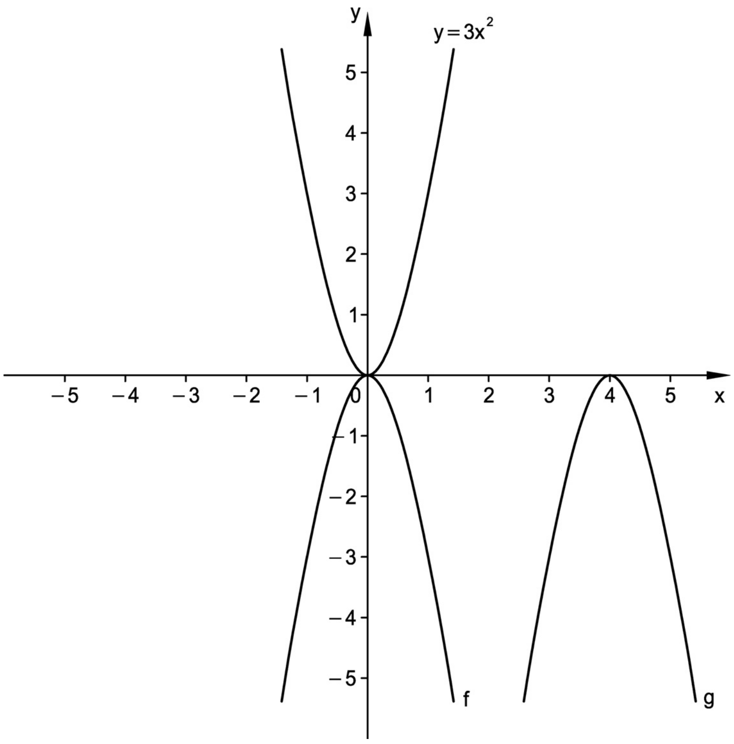
FORMATIVE ASSESSMENT

1. Identify the following graphs and explain the translation.

a)



b)



2. Draw the following graphs, each on a separate set of axes by translating $y = ax^2$ where $a > 0$.

Explain the translation you used.

a) $y = -(x + 1)^2$ b) $y = \frac{1}{3}(x - 2)^2$

c) $y = -2(x + 3)^2$

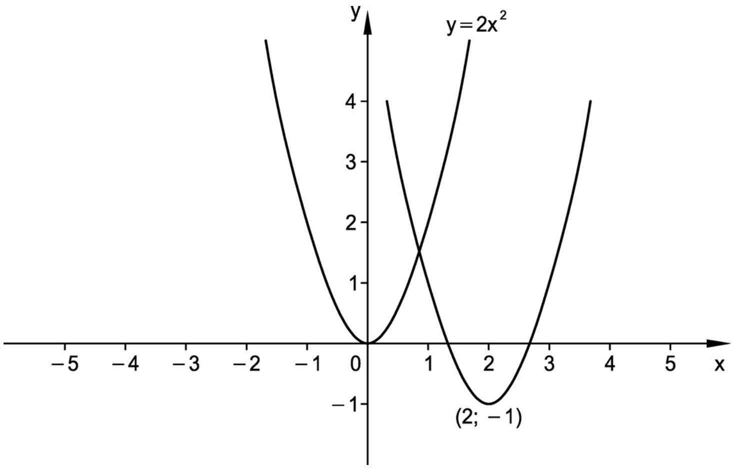
INDIVIDUAL

SUMMATIVE ASSESSMENT

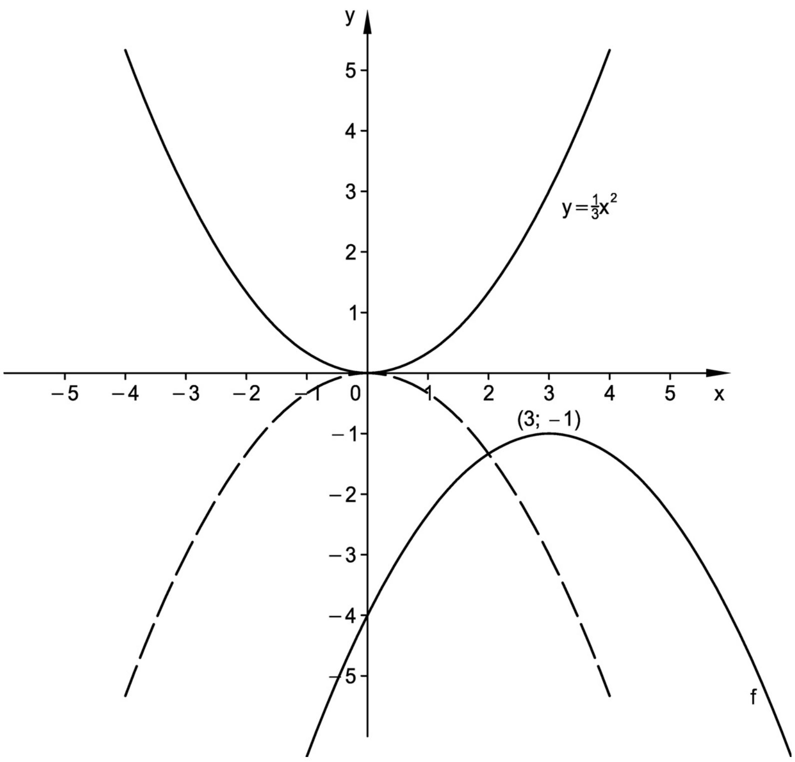
Activity 2

1. Identify the following graphs and explain the translation.

a)



b)



2. Use translation to draw the following graphs on separate sets of axes and explain the translation you need.

a) $y = -(x + 3)^2 - 1$

b) $y = \frac{1}{2}(x + 1)^2 - 2$

c) $y = -2(x - 3)^2 + 2$

Activity 3

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. Sketch the following graphs. In each case write down the translation plan. Show on the graph the asymptotes and any x- and y-intercepts.

   a) $y = \frac{3}{x-2} + 1$

   b) $y = \frac{-6}{x+1} - 2$

   c) $y = \frac{2}{x+3} - 1$

   d) $y = \frac{-4}{x-1} + 2$

2. If $y = \frac{6}{x}$ is translated 1 unit up and 3 units to the left, draw the graph and give the equation.

3. If the graph $y = \frac{12}{x}$ is reflected across the y-axis, translated 1 unit down vertically and two units to the left horizontally, write down the equation and draw the graph.

Lesson 35

TRANSLATION OF GRAPHS (2)

The exponential function and trigonometric function

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment Standard
Generate as many graphs as necessary, initially by means of point plotting, supported by available technology, to make and test conjectures about the effect of the parameters k, p, a and q for the functions including:

$y = \sin kx$	$y = \cos kx$
$y = \tan kx$	$y = \sin (x + p)$
$y = \cos (x + p)$	$y = \tan (x + p)$
$y = a(x + p)^2 + q$	
$y = abx + p + q$	$y = \frac{a}{x + p} + q$

Overview

In this lesson you will:

- Revise translating the exponential graph vertically.
- Learn to translate the exponential graph horizontally.
- Learn to translate the exponential graph both vertically and horizontally.
- Revise translating trigonometric graphs vertically.
- Learn to translate trigonometric graphs horizontally.
- Learn to draw trigonometric graphs when the period changes.

Lesson

DVD

The exponential function

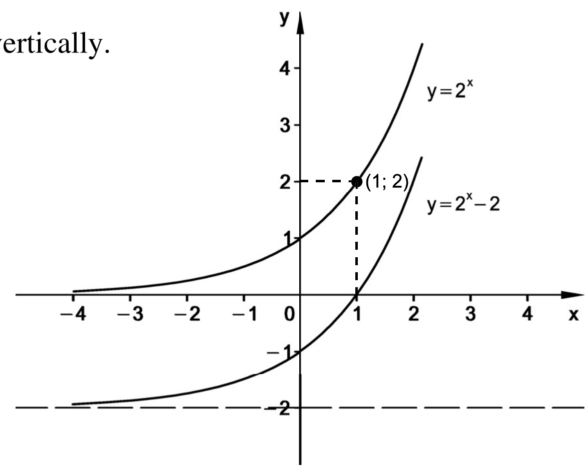
Last year you translated the graph vertically.

Let's revise it.

Draw: $y = 2^x$

Note: There is an asymptote at $y = 0$ (x–axis) so when we translate the graph, we first translate the asymptote.

Draw: $y = 2x - 2$

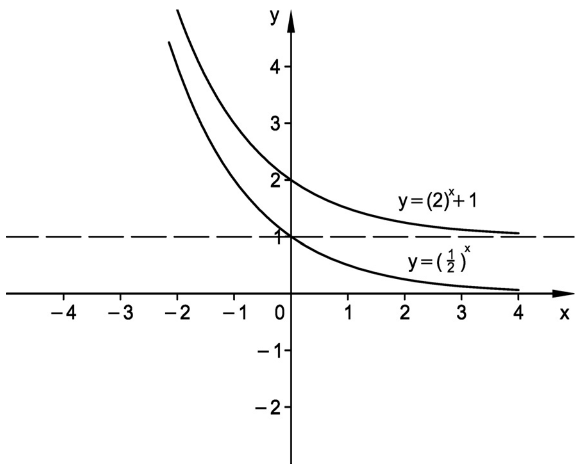


Draw: $y = \left(\frac{1}{2}\right)^x$

Note: This is the graph of $y = 2x$ reflected across the y–axis

It is a decreasing function because the gradient is negative.

Draw: $y = \left(\frac{1}{2}\right)^x + 1$ (translate the asymptote first)



Activity 1

FORMATIVE ASSESSMENT

Investigation

On the same set of axes sketch:

$y = 2(x + 1)$ $\quad y = 2(x - 1)$ $\quad y = 2(x + 3)$ $\quad y = 2(x - 4)$

You can put your calculator into table mode and select negative and positive values. If you do not have a calculator with table mode make a table of values.

x					
y					

Choose positive values and negative values and remember $2(0) = 1$ is very important.

What did you learn from this investigation?

If $y = 2(x - \text{p})$, the graph is translated p units to the left or right horizontally.

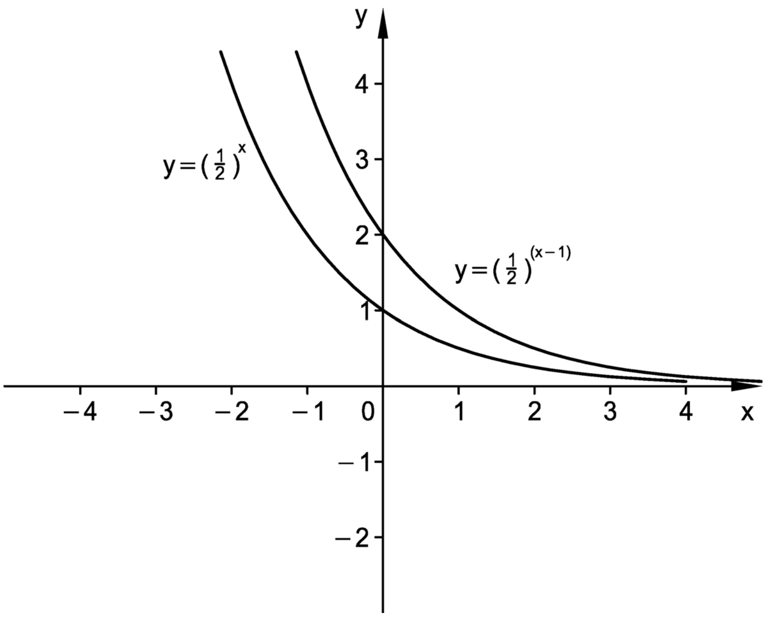
The line $y = 0$ (the x–axis) is the horizontal asymptote.

Example 1

DVD

Draw $y = \left(\frac{1}{2}\right)^{x-1}$ is translated 1 unit to the right.

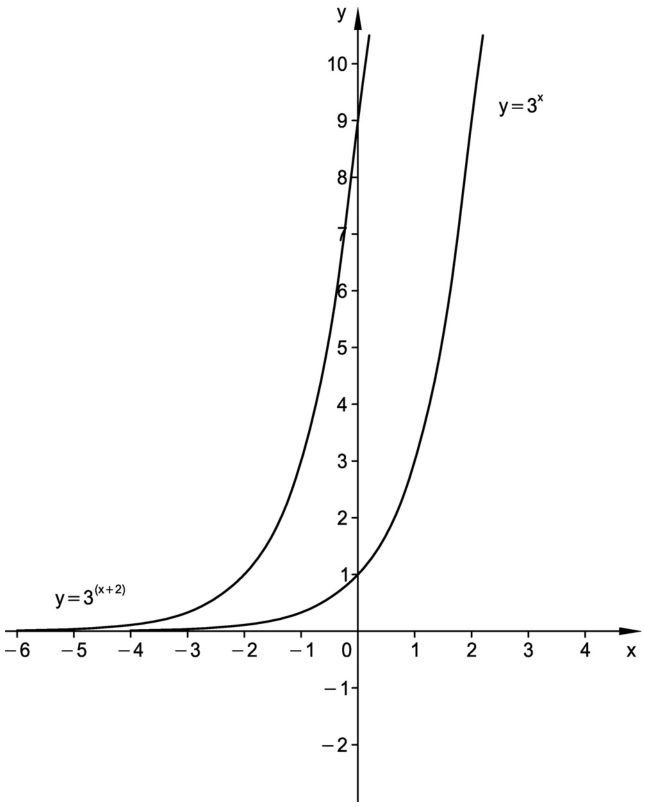
Translation: The graph $y = \left(\frac{1}{2}\right)^{x}$ is translated 1 unit to the right.



Example 2

Draw $y = 3^{x+2}$

Translation: The graph y = 3x is translated 2 units to the left.

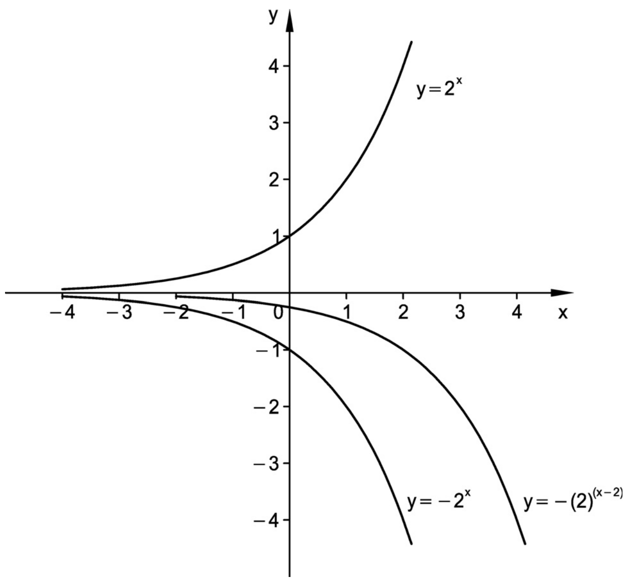


Example 3

What about $y = -2^{x-2}$

Translation: Draw $y = -\,2^{x}$

Reflect it across the x–axis to get $y = -2^{x}$

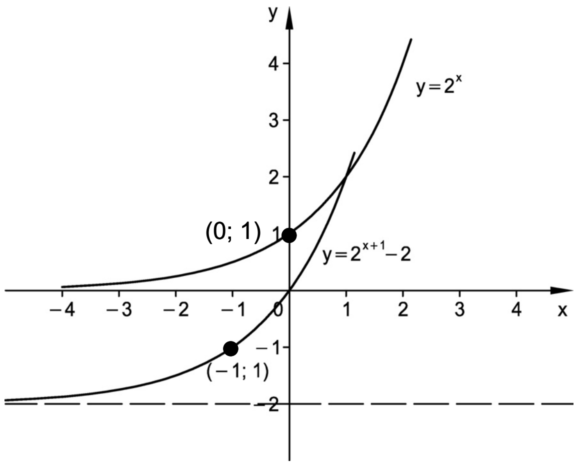


Example 4

Vertical and horizontal shift: Sketch: $y = 2x + 1 - 2$

Translation: Draw $y = 2x$

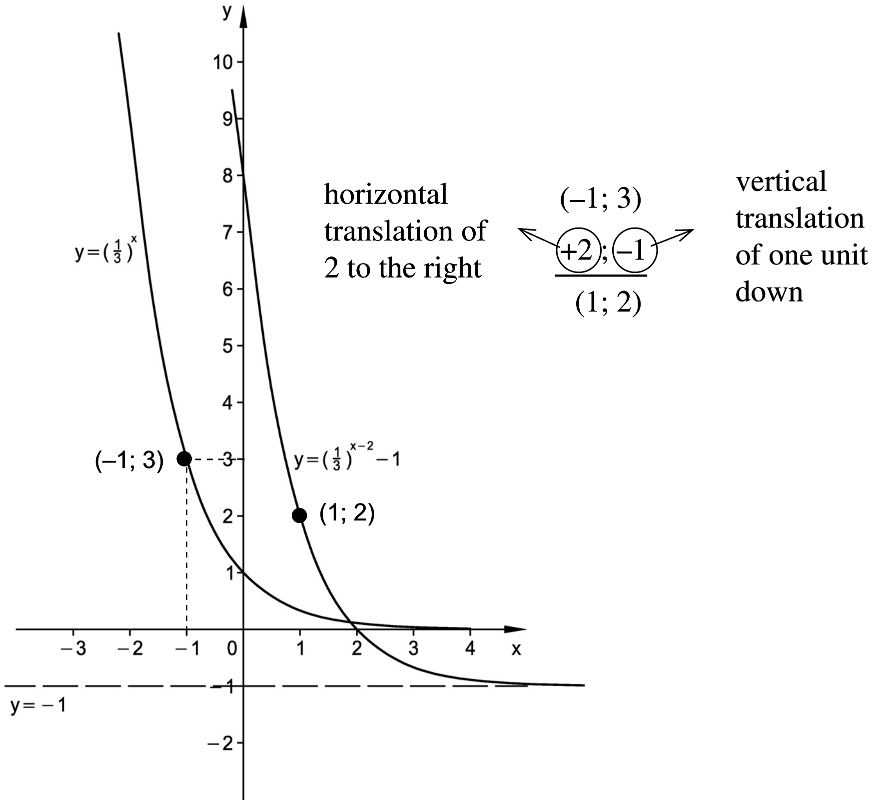
Translate it 1 unit to the left and 2 units down



Example 5

One more: Sketch $y = \left(\frac{1}{3}\right)^{x-2} - 1$

Translation: Draw $y = \left(\frac{1}{3}\right)^{x}$ translate it 2 units right and 1 unit down.



INDIVIDUAL

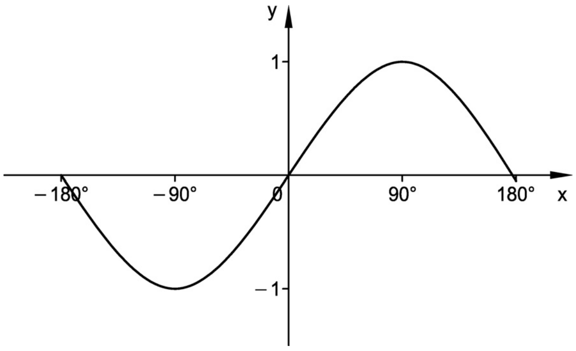
SUMMATIVE ASSESSMENT

DVD

Activity 2 no's A1 – 9

Trigonometry Graphs

Last Year

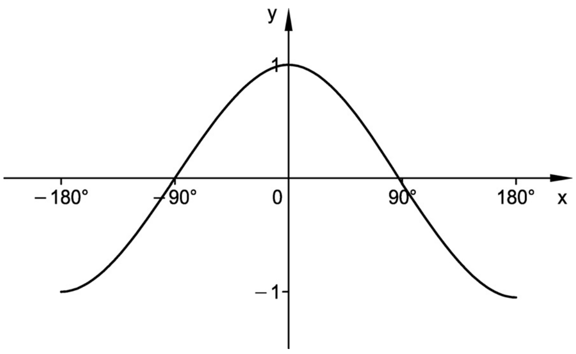


$y = \sin x$

$x \in [-180°; 180°]$

Amplitude: 1

Period: 360°

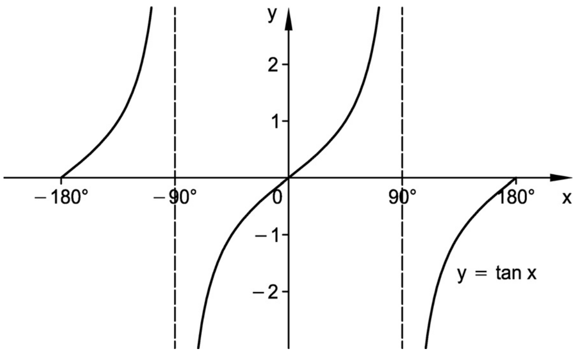


$y = \cos x$

$x \in [-180°; 180°]$

Amplitude: 1

Period: 360°



$y = \tan x$

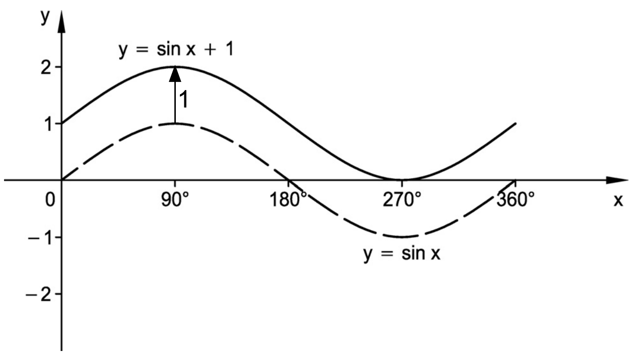
$x \in [-180°; 180°]$

Period: 360°

Asymptotes: $90° + k180°$ $k \in Z$

Vertical translation: $y = \sin x + 1$ $x \in [0° ; 360°]$

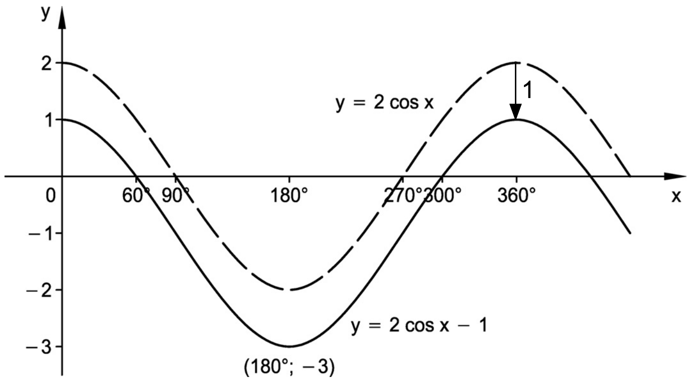
$y = \sin x$ translates up 1 unit



$y = 2 \cos x - 1$ $x \in [0° ; 360°]$

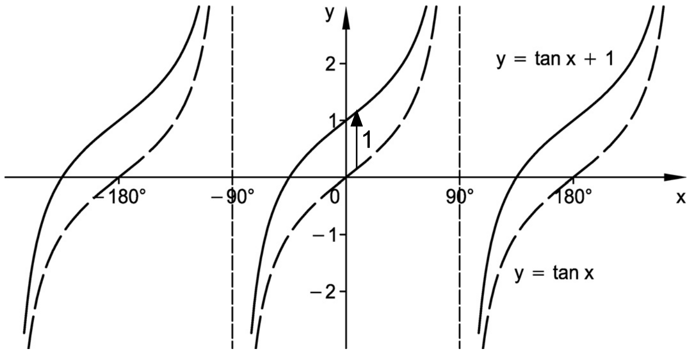
Translation $y = \cos x$ has an amplitude of 2 – stretches 2 up and 2 down.

$y - 2 \cos x$ translates 1 down.



$y = \tan x + 1 \;\; x \in [-180° ; 180°]$

Translation: $y = \tan x$ translates up 1 unit



Investigation: Use your calculator in table mode to sketch the following graphs if $x \in [0° ; 360°]$

$y = \sin 2x$	$y = \sin 3x$	$y = \sin 2x$	
$y = \cos 2x$	$y = \cos 2x$	$y = \cos 4x$	if $x \in [0° ; 360°]$
$y = \tan 2x$	$y = \tan 3x$	$y = \tan 4x$	if $x \in [0° ; 90°]$

What happened to the graphs?

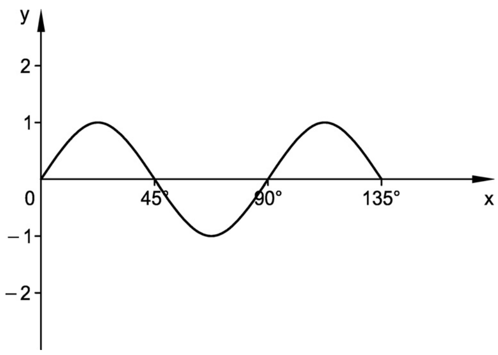
They either shrunk or stretched horizontally.

We say that the period of the graph changed.

For $y = \sin bx$ and $y = \cos bx$ the period is $\frac{360°}{b}$

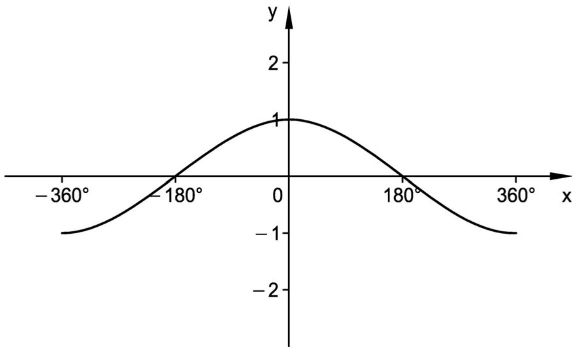
For $y = \tan bx$ the period is $\frac{180°}{b}$

Identify the following graphs:

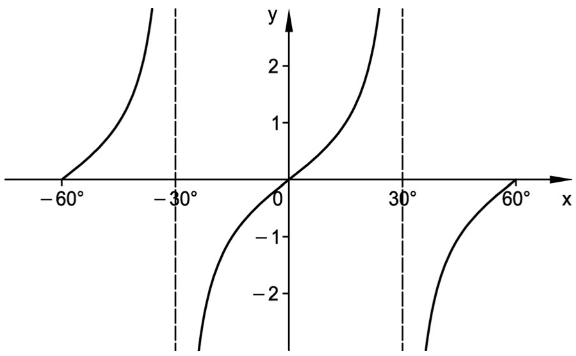


$y = \sin 4x \qquad$ if $x \in [0° ; 135°]$

LC DVD



$y = \tan 3x$ if $x \in [-360° ; 360°]$

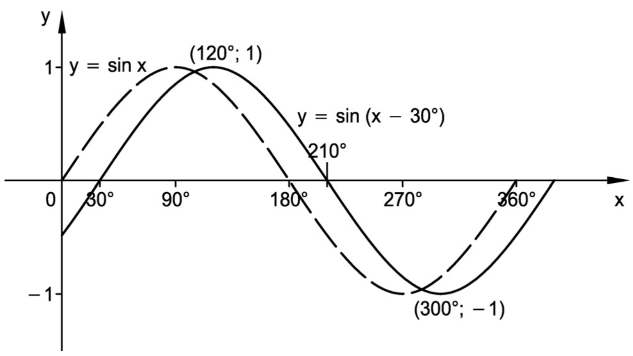


$y = \tan 3x$ if $x \in [-60° ; 60°]$

Horizontal Shift

$y = \sin(x - 30°)\ x \in [0° ; 360°]$

Translation: $y = \sin x$ translates 30° to the right.



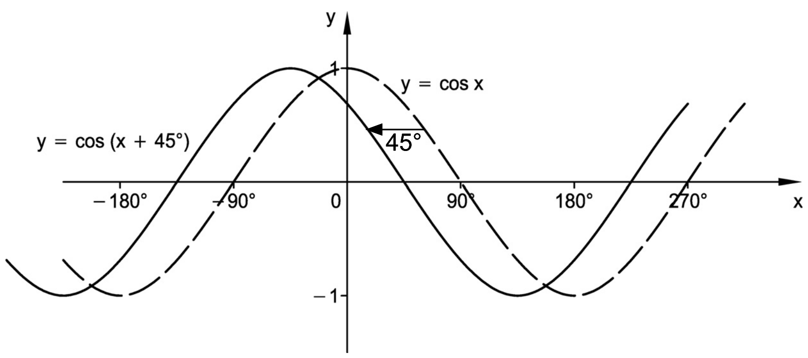
y–intercept $x = 0°$ $\sin(-30°) = -\frac{1}{2}$

Activity 2 A to B, C, D – 12

The following cos and tan graphs are additional examples for you to work through.

Sketch: $y = \cos(x + 45°)\ x \in [-180° ; 180°]$

Translation: $y = \cos x$ translates 45° to the left.

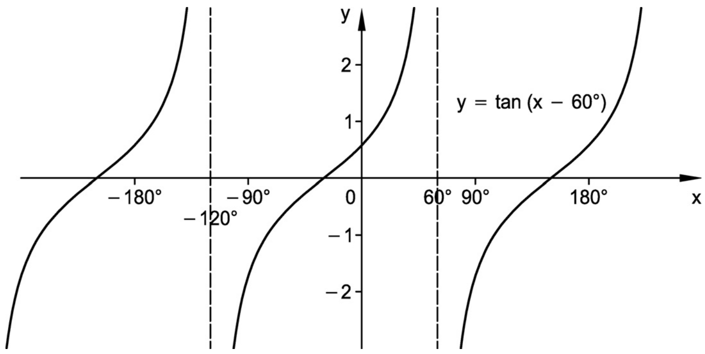


$y = \cos(x + 45°)$

$y = \cos x$

Sketch: $y = \cos(x - 60°)$ $x \in [-0° \,;\, 180°]$

Translation: $y = \tan x$ translates 60° to the right.



$y = \tan(x - 60°)$

Activity 1

PAIRS

FORMATIVE ASSESSMENT

1. a) Sketch $y = 3x$
   b) Reflect $y = 3x$ across the y–axis and write down the equation.
   c) Reflect $y = 3x$ across the x–axis and write down the equation.
2. a) Sketch $y = \frac{1}{4}x$
   b) Reflect $y = \frac{1}{4}x$ across the y–axis and write down the equation.
   c) Reflect $y = \frac{1}{4}x$ across the x–axis and write down the equation.

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

A. Sketch each of the following graphs on a separate set of axes and describe the translation you used.

1. $y = 3x + 1$
2. $y = \left(\frac{1}{2}\right)^x - 2$
3. $y = -2x - 1$
4. $y = 2x - 2$
5. $y = \left(\frac{1}{3}\right)^{x-1}$
6. $y = 2x + 1 - 2$
7. $y = \left(\frac{1}{3}\right)^{x-1} - 2$
8. $y = -2x + 1 - 1$
9. $y = \left(\frac{1}{3}\right)^{x-1} + 1$
10. $y = \sin(x + 30°)$ if $x \in [0 \,;\, 360°]$
11. $y = 2\cos(x - 45°)$ if $x \in [0 \,;\, 360°]$
12. $y = \tan(\, x + 60°)$ if $x \in [0° \,;\, 360°]$

B. 1. Sketch $f(x) = 2x + 1 - 1$

2. Sketch g if g is the reflection of f across the y–axis.

3. Sketch h if h is the reflection of f across the x–axis.

C. 1. Sketch $y = \cos(x + 30°)$ if $x \in [-180° ; 180°]$

2. If the y–axis is translated 60° to the right, give the equation of the new graph.

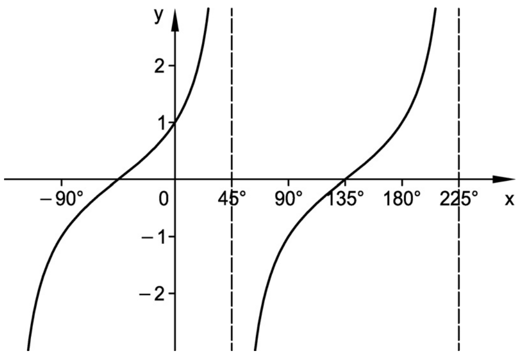
3. If the graph is translated 2 units vertically down, give the equation of the new graph.

4. If the graph is translated horizontally 20° to the left, find the new equation of the graph.

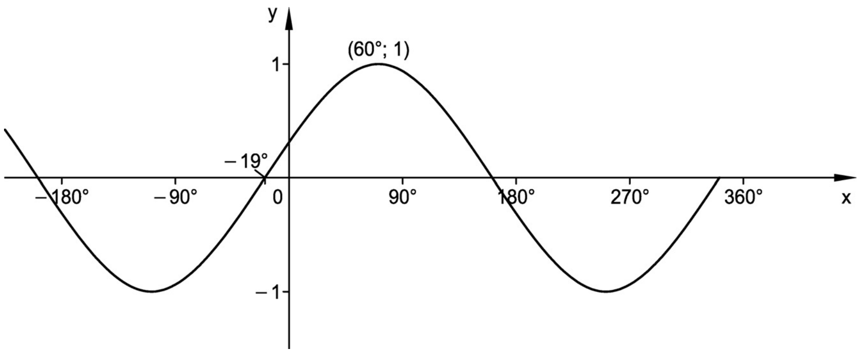
D. 1. Sketch $y = \sin(x - 60°)$ if $x \in [-180° ; 360°]$

2. Name this graph in 3 different ways.

E. Give an equation for the following graph.



F.



1. Find the equation of g.

2. Find the co-ordinates of A.

3. Find another equation for g.

FINDINGS EQUATIONS OF FUNCTIONS

Lesson 36

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment Standard
Generate as many graphs as necessary, initially by means of point plotting, supported by available technology, to make and test conjectures about the effect of the parameters k, p, a and q for the functions including:

$y = \sin kx$	$y = \cos kx$
$y = \tan kx$	$y = \sin (x + p)$
$y = \cos (x + p)$	$y = \tan (x + p)$
$y = a(x + p)^2 + q$	
$y = abx + p + q$	$y = \frac{a}{x + p} + q$

Lesson for overview

In this lesson you will:

- Find the equation of parabola by using the turning point.
- Find the equation of hyperbola by identifying the asymptotes and substituting a point.
- Find the equation of an exponential graph by identifying the asymptote and substituting a point.

Lesson

LC DVD

1. The parabola

Equation: $y = a(x - p)^2 + q$

Remember the turning point is $(p\ ;\ q)$

To find the equation we need:

(1) Turning point

(2) One other point to get the value of a

Example 1

Find the equation of the following parabola

Solution

Step 1: Fill in the turning point

$y = a(x - 2)^2 - 3$

Step 2: Use the other point on the graph to substitute

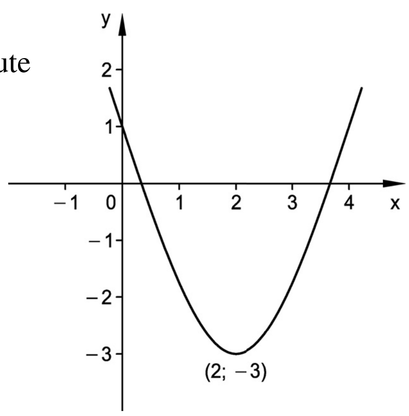
$x = 0 \quad y = 1$

$1 = a(-2)^2 - 3$

$4 = 4a$

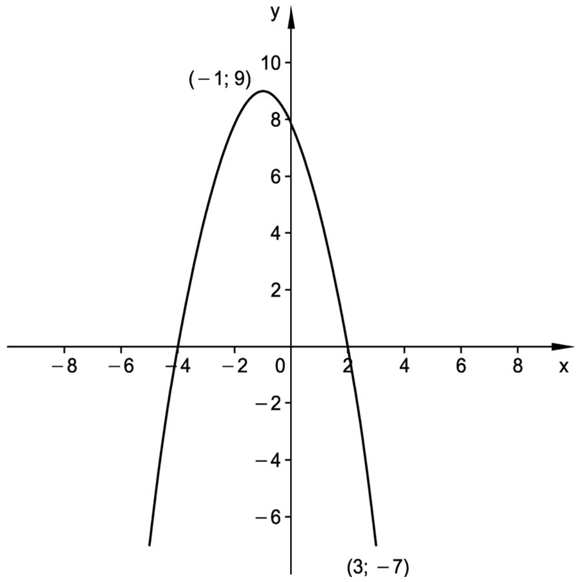
$a = 1$

Equation: $y = a(x - 2)^2 - 3$



Example 2

a) Find the equation of the following parabola and write it in the form $y = ax^2 + bx + c$



b) Find the x and y intercepts

Put in the turning point

$y = a(x + 1)^2 + 9$

Substitute the point (3 ; –7)

$x = 3 \quad y = –7$

$–7 = a(4)^2 + 9$

$–16 = 16a$

$a = –1$

Equation: $y = –(x + 1)^2 + 9$

$y = –(x^2 + 2x + 1) + 9$

$y = –x^2 – 2x + 8$

(0 ; 8)

y–intercept $x = 0$

x–intercept $y = 0$

$0 = –x^2 – 2x + 8$

$x^2 + 2x – 8 = 0$

$(x + 4)(x – 2 = 0$

x–intercepts (–4 ; 0)(2 ; 0)

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

DVD

2. Hyperbola

Form of equation: $y = \frac{k}{x - p} + q$

Remember $x = p$ *and* $y = q$ *are asymptotes*

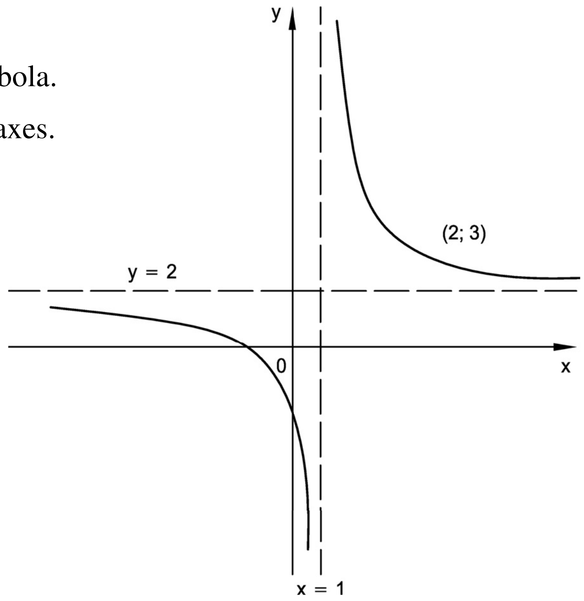
First fill in the asymptotes

Substitute a point to find k

Example 1

a) Find the equation of the parabola.

b) Find any intercepts with the axes.



Put in the asymptotes first

$y = \frac{k}{x-1} + 2$

Substitute point (2 ; 3) to find k

$3 = \frac{k}{2-1} + 2$

$3 = k + 2$

$k = 1$

so

$y = \frac{1}{x-1} + 2$

x–intercept $y = 0$

$0 = \frac{1}{x-1} + 2$

$-2 = \frac{1}{x-1}$

$-2x + 2 = 1$

$-2x + 2 = 1$

$-2x = -1$

$x = \frac{1}{2}$

x–intercepts $\left(\frac{1}{2} ; 0\right)$

y–intercept $x = 0$

$y = \frac{1}{-1} + 2$

$y = 1$

y intercept (0 ; 1)

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

DVD

Exponential graphs

Equation: $y = a^{x+p} + q$

$y = q$ *is a horizontal asymptote*

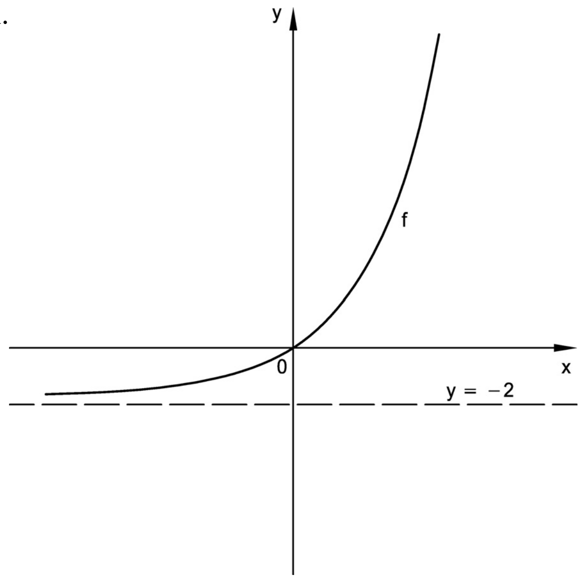
If a > 1 the function is increasing

If 0 < a < 1 the function is decreasing

To find the equation first put in the asymptote and then substitute the point.

Example 1

If f : $x \rightarrow 2^{x+p} + q$ find the equation.



Solution

$y = 2^{x+p} - 2$

Substitute (0 ; 0) $x = 0$ $y = 0$

$0 = 20 + p - 2$

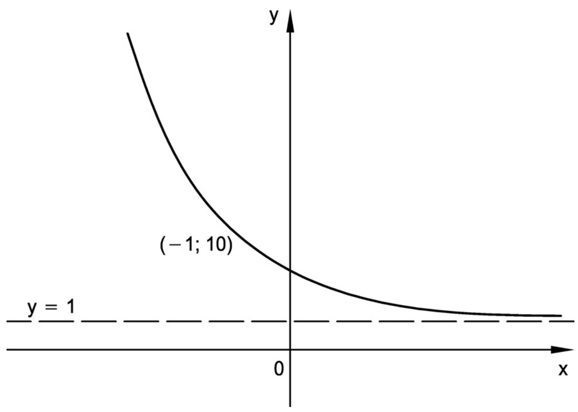
$2 = 2p$

$p = 1$

$y = 2x + 1 - 2$

Example 2

Find the equation of f if $f(x) = \left(\frac{1}{3}\right)^{x+p} + 1$ and find the y–intercept



Solution

$y = \left(\frac{1}{3}\right)^{x+p} + 1$

Substitute $x = -1$ $y = 10$

$10 = \left(\frac{1}{3}\right)^{-1+p} + 1$

$9 = \left(\frac{1}{3}\right)^{-1+p}$

Make the base the same

$32 = (3–1)–1 + p$

$32 = 31 - p$

$2 = 1 - p$

$p = -1$

$y = \left(\frac{1}{3}\right)^{x+p} + 1$

y–intercept make $x = 0$

$y = \left(\frac{1}{3}\right)^{-1} + 1$

$y = (3–1)–1 + 1$

$y = 4$

y–intercepts (0 ; 4)

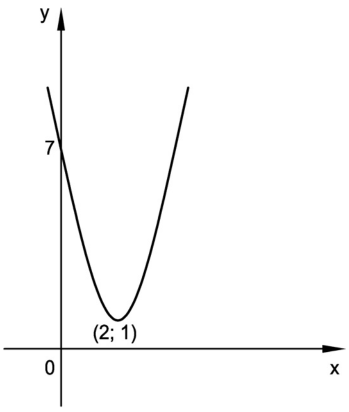
Activity 1

INDIVIDUAL

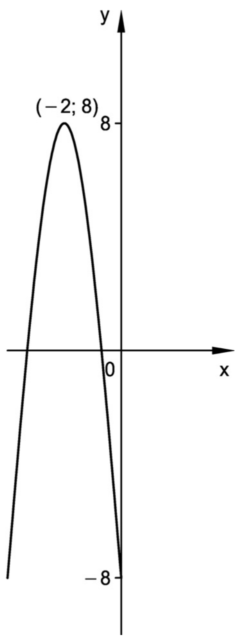
SUMMATIVE ASSESSMENT

1, Find the equations of the following parabolas

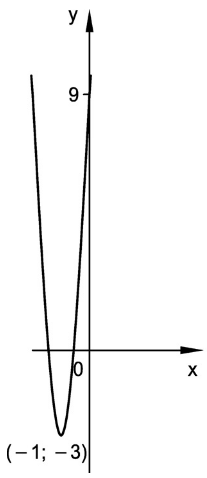
(a)



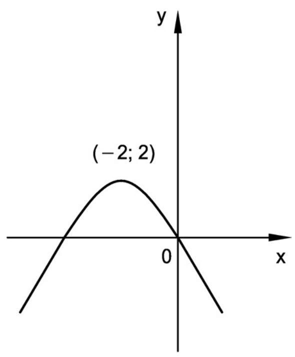
(b)



(c)

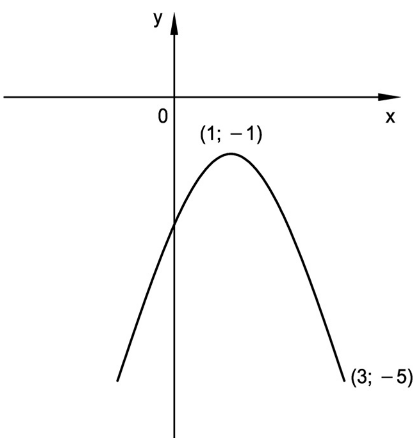


(d)



2. a) Find the equation of the following parabola

b) If the graph is translated 4 units up and 4 units to the left, write down the equation of the function.



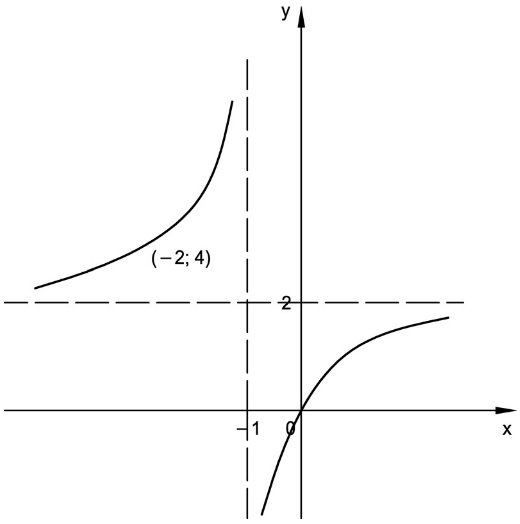
INDIVIDUAL

SUMMATIVE ASSESSMENT

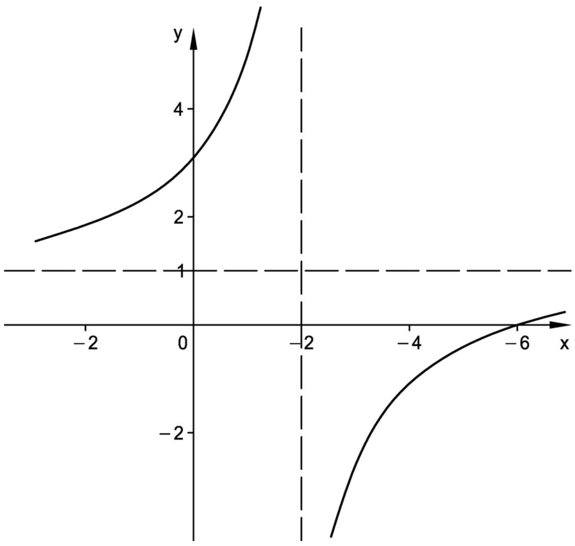
Activity 2

Find the equations of each of the following hyperbola and if necessary, write down the *x* and *y*–intercepts.

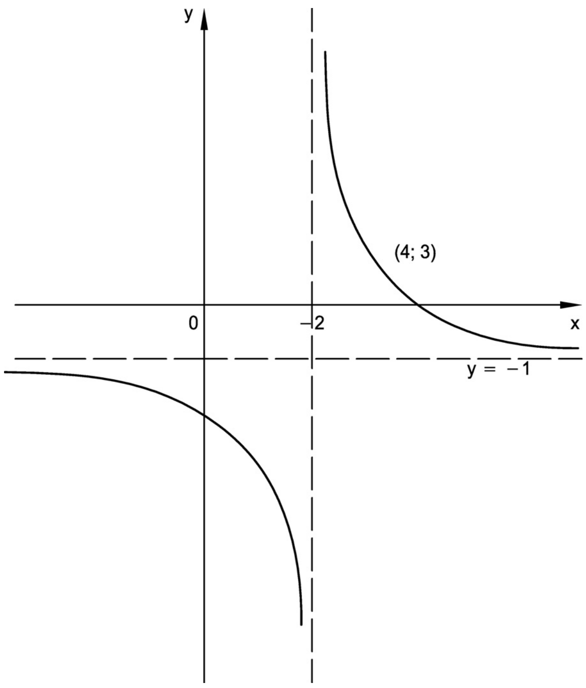
1.



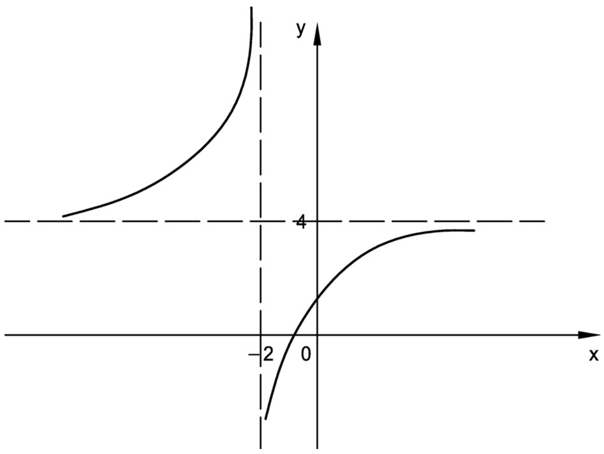
2.



3.



4.



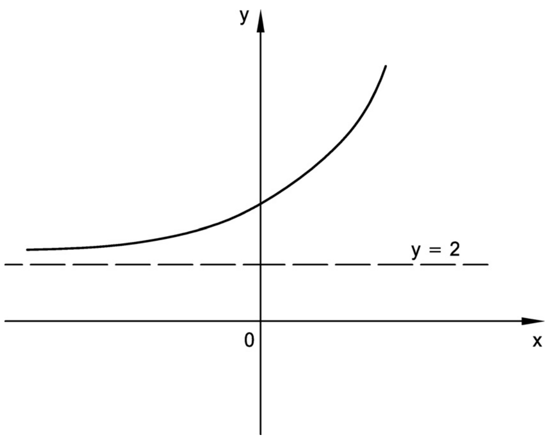
Activity 3

INDIVIDUAL

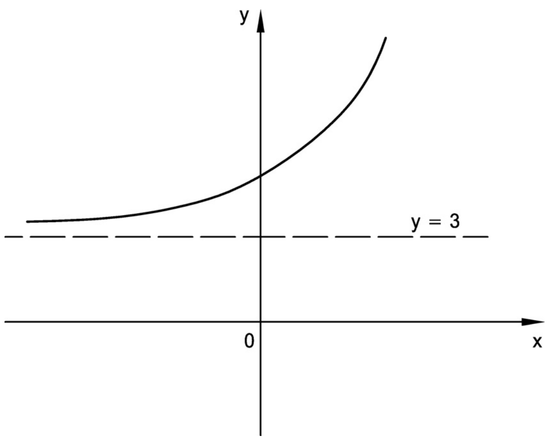
SUMMATIVE ASSESSMENT

In each of the following graphs, find the value(s) of p and q.

1.

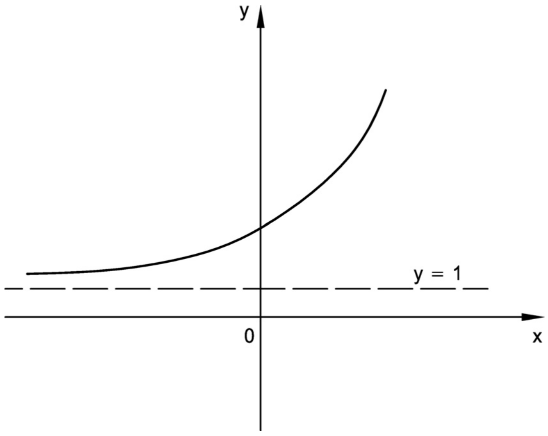


2.



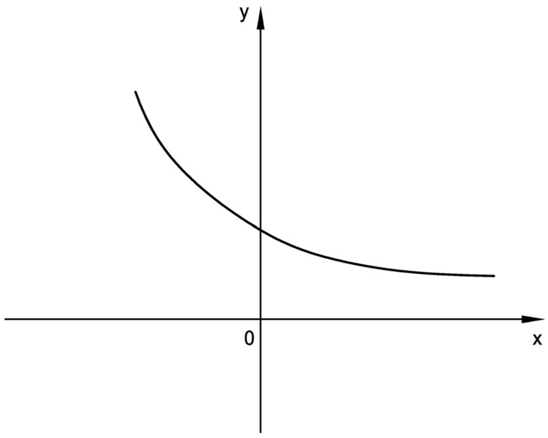
Find the y–intercept if $y = 2^{x+p} + q$

3.



Find the equation and the y–intercept.

4. If $y = a.b^x$, find the value of a and b.



TRANSLATION OF GRAPHS

Drawing parabolas

Lesson 37

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment Standard
Generate as many graphs as necessary, initially by means of point plotting, supported by available technology, to make and test conjectures about the effect of the parameters k, p, a and q for the functions including:

$y = \sin kx$	$y = \cos kx$
$y = \tan kx$	$y = \sin (x + p)$
$y = \cos (x + p)$	$y = \tan (x + p)$
$y = a(x + p)^2 + q$	
$y = abx + p + q$	$y = \frac{a}{x + p} + q$

Overview

In this lesson you will:

- Draw the parabola by completing the square.
- Draw the parabola by a suitable formula.
- Use substitution to draw the parabola.
- Find the equation of a parabola when given x-intercepts.

Lesson

LC DVD

$y = ax^2 + bx + c$ is the graph of a quadratic function

To draw the graph we need to write the equation in the form $y = a(x – p)^2 + q$ by completing the square

a tells us the shape

If $a < 0$ ☹ sad. If $a > 0$ ☺ happy.

$(p ; q)$ is the turning point.

y–intercept make $x = 0$

You can use symmetry to find another point.

To find the x-intercepts (if possible) make $y = 0$.

Examples

1. if $y = x^2 – 6x + 8$
   a. Find the turning point
   b. Find the y-intercepts
   c. Find the x-intercepts
   d. Draw the graph

Solutions

a. We need to complete the square

$y = x^2 – 6x \quad +8$

Add and subtract half the co-efficient of x

$y = x^2 – 6x + (–3)^2 – 9 + 8$

Factorise the square trinomial

$y = (x - 3)^2 - 1$

Turning point $(3\ ;\ -1)$ Shape 

b. y-intercepts $x = 0$

$(0\ ;\ 8)$

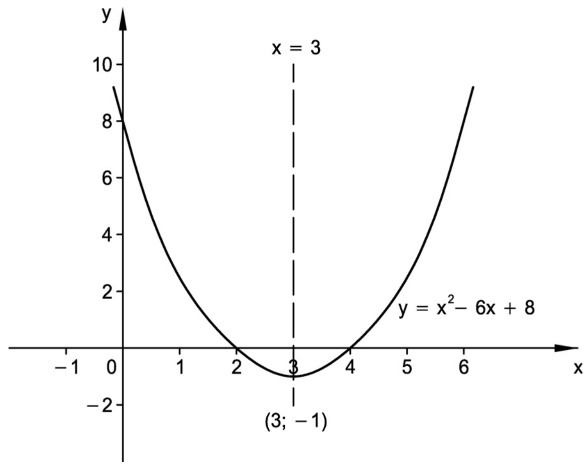
c. x-intercepts $y = 0$

$x^2 - 6x + 8 = 0$ OR $(x - 3)^2 - 1 = 0$

$(x - 4)(x - 2) = 0$ $x - 3 = \pm 1$

$(4\ ;\ 0)\ (2\ ;\ 0)$ $x = 3 \pm 1 \Rightarrow x = 4\ \alpha\ x = 2$

d. Graph



$y = x^2 - 6x + 8$

PAIRS

FORMATIVE ASSESSMENT

Activity 1

Let's get a formula

This is what we did each time $y = ax^2 + bx + c$

Divide by a $\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$

Complete the square $\frac{y}{a} = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$

$\frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$

Put a back $y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

So the turning point is: $\left(-\frac{b}{2a}\ ;\ c - \frac{b^2}{4a}\right)$

$x = \frac{-b}{2a}$ is the axis of symmetry

$\frac{4ac - b^2}{4a}$ is the maximum value if $a < 0$

$\frac{4ac - b^2}{4a}$ is the minimum value if $a > 0$

2. We will use the formula to draw the graph.

Sketch $y = -x^2 - 4x + 12$ and show the turning point and intercepts with the axes.

Solution

Shape  Turning point: $\left(-\frac{b}{2a} ; \frac{4ac - b^2}{4a}\right)$

$a = -1$ $b = -4$ $c = 12$

Turning point: $\left(\frac{-(-4)}{-2} ; \frac{4(-1)(12) - 16}{-4}\right)$

$(-2 ; 18)$

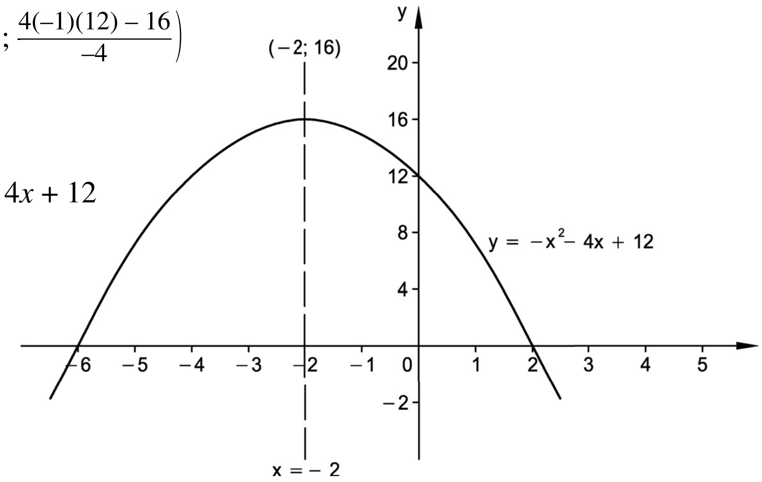
y-intercepts $(0 ; 12)$

x-intercepts $0 = -x^2 - 4x + 12$

$x^2 + 4x - 12 = 0$

$(x + 6)(x - 2) = 0$

$(-6 ; 0)(2 ; 0)$



3. Sketch $y = 2x^2 + x - 6$

Solution

Shape 

Axis of symmetry: $x = -\frac{b}{2a}$

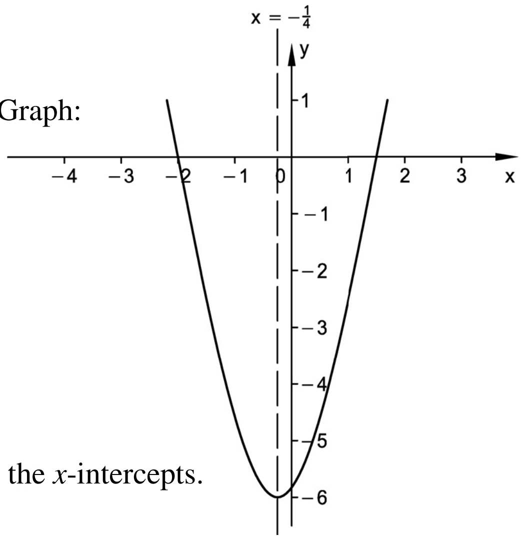
$x = -\frac{1}{4}$

Now instead of using the formula for the minimum value substitute into the original equation.

$y = -\frac{49}{8}$

$y = -6\left(\frac{1}{8}\right)$

Turning point: $\left(-\frac{1}{4} ; -6\left(\frac{1}{8}\right)\right)$ Graph:



y-intercepts $(0 ; -6)$

x-intercepts $0 = 2x^2 + x - 6$

$0 = (2x - 3)(x + 2)$

$\left(\frac{3}{2} ; 0\right)(-2 ; 0)$

Finding the equation when you are given the x-intercepts.

Example

$y = a(x - 1)(x + 6)$

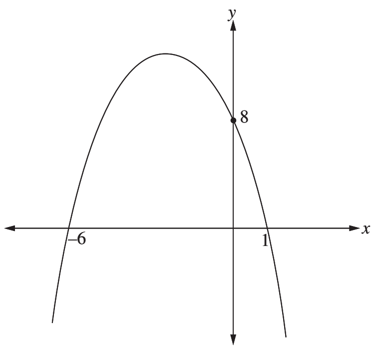
Then $8 = a(-1)(6)$

$8 = -6a$

$\therefore\ a = -\frac{3}{4}$

So: $y = -\frac{3}{4}(x^2 + 5x - 6)$

$y = -\frac{3x^2}{4} - \frac{15}{4}x + \frac{9}{2}$



DVD

PAIRS

SUMMATIVE ASSESSMENT

Activity 1

Sketch each of the following graphs by completing the square to find the turning point and find the y-intercepts and the x-intercepts if possible.

1. $y = 2x^2 - 7x - 4$
2. $y = -x^2 + 8x - 12$
3. $y = -x^2 + 2x - 6$

INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 2

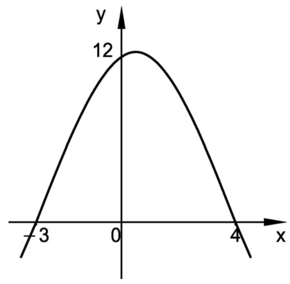
1. Sketch the following functions:

   a) $f : x \rightarrow (-x^2 + 3x - 2$ b) $g : x \rightarrow (2x + 1)(x - 3)$

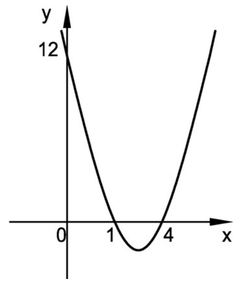
   c) $h : x \rightarrow -2x^2 + 5x$

2. Find the equations of the following functions:

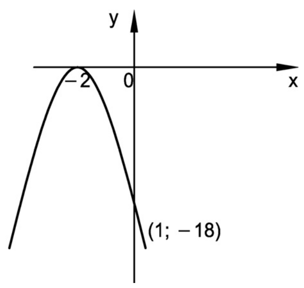
   a)



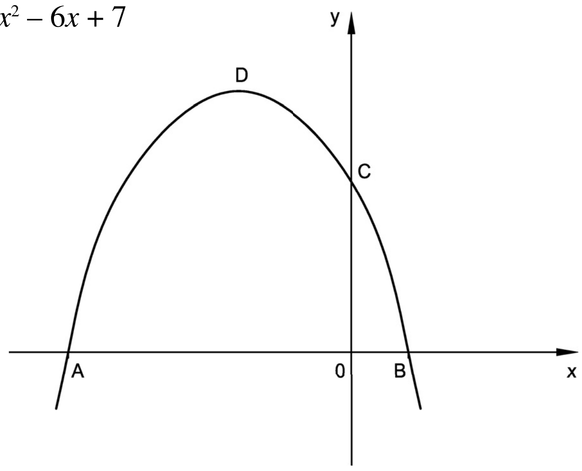
   b)



   c)



3. This is the graph of $f ; x \rightarrow -x^2 - 6x + 7$



   a) Find the co-ordinates of A, B, C and D (the turning point)

   b) If $-x^2 - 6x + k = 0$ has no real roots find k.

   c) If $-x^2 - 6x + k = 0$ has two negative roots find the possible values of k.

4. a) Show that $2p^2x^2 - 2px + 1$ is always positive if $p > 0$

   b) Sketch $y = 2p^2x^2 - 2px + 1$ and write down the co-ordinates of the turning point in term of p if $p > 0$

APPLICATION OF GRAPHS

Lesson 38

Learning outcomes and assessment standards

Learning Outcome 2: Functions and Algebra
Assessment Standard
Generate as many graphs as necessary, initially by means of point plotting, supported by available technology, to make and test conjectures about the effect of the parameters k, p, a and q for the functions including:

$y = \sin kx$	$y = \cos kx$
$y = \tan kx$	$y = \sin(x + p)$
$y = \cos(x + p)$	$y = \tan(x + p)$
$y = a(x + p)^2 + q$	
$y = abx + p + q$	$y = \frac{a}{x + p} + q$

Overview

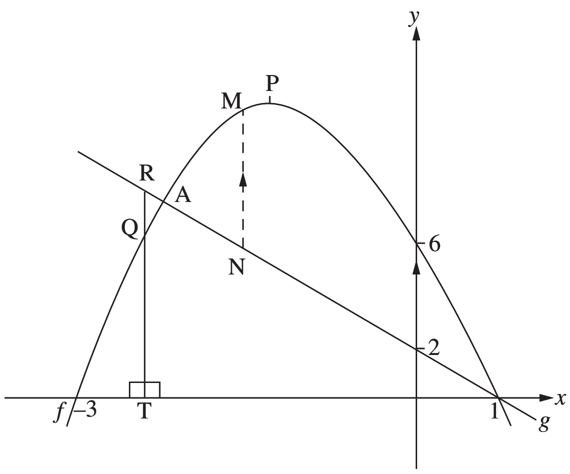
In this lesson you will:

- Analyse graphs.
- Apply the graphs you have learnt to real life problems.

Lesson

Example 1

Look at this graph



1. Find the equations of f and g

   For f put in the x-intercepts

   $y = a(x + 3)(x - 1)$

   $y = a(x^2 + 2x - 3)$

   Substitute pt(0; 6)

   $6 = a(-3)$

   $a = -2$

   $f(x) = -2x^2 - 4x + 6$

   For the straight line we use $y = mx + c$

   $c = 2 \qquad m = -2$

   $g(x) = -2x + 2$

2. Find the coordinates of the turning point P.

   axis of symmetry $x = \frac{-b}{2a}$

   $f(x) = -2x^2 - 4x + 6$

   $x = \frac{-(-4)}{2(-2)}$

   $x = 1$

   Substitute into f to find the maximum value.

   $f(-1) = -2(1) - 4(-1) + 6$

   $= 8$

   P(–1; 8)

3. Find the coordinates of A. At A the two graphs intersect.

$-2x + 2 = -2x^2 - 4x + 6$

$2x^2 + 2x - 4 = 0$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$

$y = 6$

A(–2; 6)

4. If OT = $2\frac{1}{2}$ units, find the length of RQ.

RQ = Top graph – bottom graph

RQ $= -2x + 2 -(-2x^2 - 4x + 6)$

RQ $= -2x + 2 + 2x^2 + 4x - 6$

RQ $= 2x^2 + 2x - 4$

Now substitute $x = -\frac{5}{2}$

RQ = 3,5 units

5. If MN is a variable line parallel to the y-axis between A and B, find the maximum length of MN.

MN = Top – Bottom

MN $= -2x^2 - 4x + 6 - (-2x + 2)$

MN $= -2x^2 - 4x + 6 + 2x - 2$

MN $= -2x^2 - 2x + 4$

We need the maximum value of this parabola.

Remember: Turning point is $\left(-\frac{b}{2a}; \frac{4ac - b^2}{4a}\right)$

so $\frac{4ac - b^2}{4a}$ is the maximum value

$a = -2 \quad b = -2 \quad c = 4$

Maximum value is $\frac{-36}{-8} = 4\frac{1}{2}$

Activities 1–6

INDIVIDUAL

SUMMATIVE ASSESSMENT

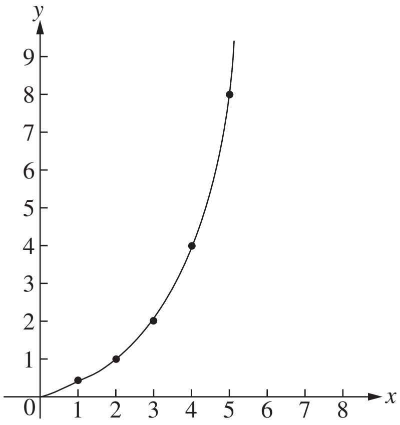
Example 2

A nutritionist measures the mass of a specimen of yeast every hour.

His observations are given in the following table.

Hours since 1st observation (x)	Mass of yeast in m.g. (y)
1	$\frac{1}{2}$
2	1
3	2
4	4
5	
6	

a) Plot the points of the graph and join them with a smooth curve.



b) Give an equation for the graph.

This is a geometric sequence.

$T_n = ar^{n-1}$

$f(x) = \frac{1}{2}(2)^{x-1}$

c) Interpret the graph in your own words.

- The initial mass of the yeast is $\frac{1}{4}$ m.g.
- After each hour the mass of the yeast doubles.

d) How will the original equation change if the graph is moved 3 units upwards and explain it in your own words?

Graph $y = \left(\frac{1}{2}\right)(2)^{x-1} + 3$

- The initial mass will be $3\frac{1}{4}$ m.g.
- After 1 hour the mass will be $3\frac{1}{2}$ m.g.

Example 3

Grant lets a house for R4 000 per month to a group of students.

a) If the number of students is x, how much rent must each student pay?

$\frac{4\,000}{x}$

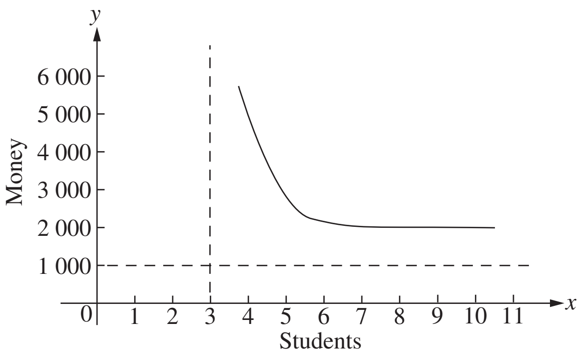
b) A fixed monthly levy of R1 000 must be paid. Write down the new equation.

$y = \frac{4\,000}{x} + 1\,000$

c) If, for some reason, three of the students can't pay, write down the new equation.

$y = \frac{4\,000}{x-3} + 1\,000$

d) Draw the graph



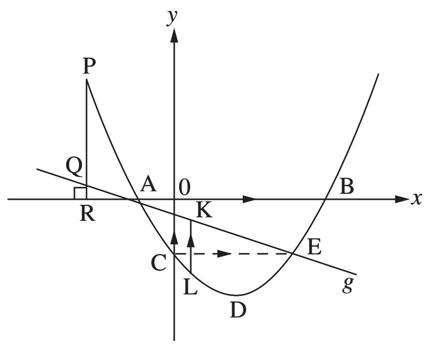
INDIVIDUAL

SUMMATIVE ASSESSMENT

Activities

1. $f(x) = 2(x - 1)^2 - 18$

   $g(x) = mx + k$



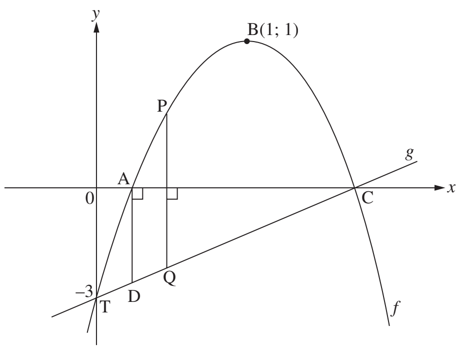
   a) Find the coordinates of A; B; C; D (turning point) and E if CE||AB.

   b) Find the values of m and k.

   c) Find the length of PQ if OR = 3 units.

   d) Find the maximum length of KL if KL||y-axis and KL is a variable line between A and E.

2.



   a) Find $f(x)$ and $g(x)$.

   b) For which values of x if $f(x) > 0$

   c) For which values of x if $f(x) > g(x)$

   d) Find the length of AD

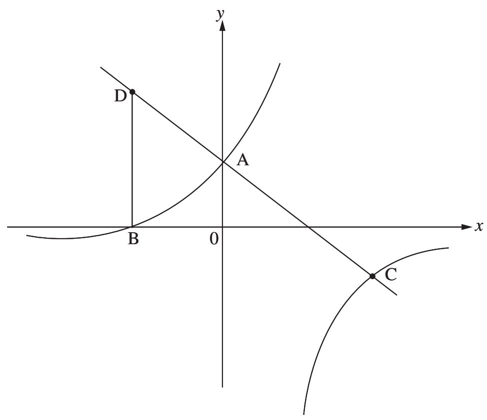
   e) Find the maximum length of PQ if PQ $\perp$ x-axis between T and C.

3. Draw the graph of $y = x^2 - 2x - 3$.

   By drawing a straight line graph, show **graphically** by using the letters P and Q where you would solve the equation $x^2 - x - 5 = 0$.

4. $f(x) = \dfrac{-4}{x-2} - 1$ $\qquad g(x) = mx + k$

   Calculate



   a) The asymptotes of the hyperbola.

   b) The coordinates of A and B.

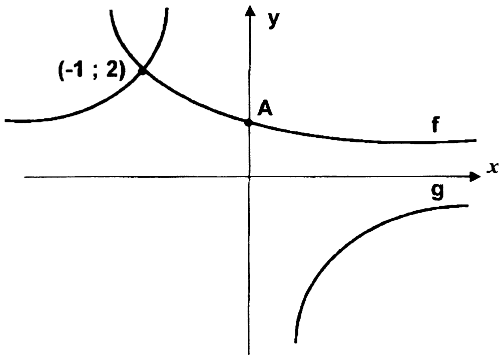
   c) The values of m and k if BD = 5 units.

   d) The coordinates of C.

   e) The distance DC.

   f) The axis of symmetry of the hyperbola.

5. $f(x) = k^x$ $\quad g(x) = \frac{a}{x}$



a) Find the values of k and a.

b) Find the coordinates of A.

c) Find the equation of the reflection of $f(x)$ across the x-axis.

d) Find the equation of the reflection of $f(x)$ across the y-axis.

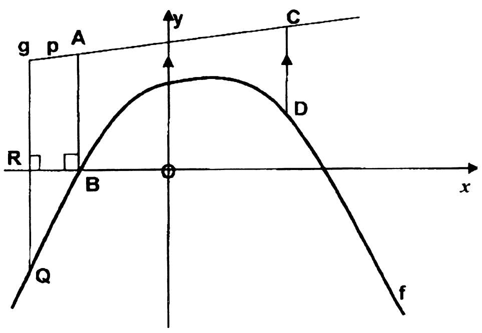
e) Find the equation of the reflection of $g(x)$ across the y-axis.

f) Find the equation of the reflection of $g(x)$ across the x-axis.

g) Find the equation of $h(x)$ if $h(x)$ is $g(x)$ moved 2 units down and 1 unit to the left.

h) Find the axis of symmetry of h.

6. The sketch represents the graphs of $f(x) = -x^2 + 2x + 3$ and $g(x) = x + 4$.



a) Find the length of AB.

b) If PQ = 7 units, find the length of OR.

c) If CD is a variable line, parallel to the y-axis, find the minimum value of CD.

Problem solving using graphs and properties of graphs.

7. The equation $M = 1{,}4(1{,}2)^x$ can be used to estimate the mass of a baby baboon up to 6 months.

M is the mass in kg.

x is the age in months.

a) Sketch the graph (you may want to use your calculator in table mode).

b) Show on your graph where you would estimate the mass of the baboon at the age of $4\frac{1}{2}$ months.

c) Use your calculator to find the correct answer to (b).

d) At what age would the baboon weigh 3,5 kg.

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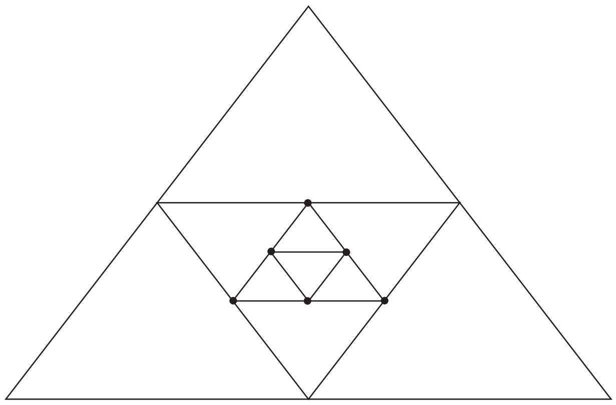
e) If another baby baboon weighed 1,8 kg at birth and its rate of growth is the same, determine the equation to suit this data.

8. This diagram shows an equilateral △ with sides 12 cm. The mid-points of the sides are joined to form another △ and so on.



a) Write down a formula for Pn = … where Pn is the perimeter and n is the number of the triangle.

b) Draw the graph with n along the x-axis and Pn on the y-axis.

c) What do you think eventually happens?

9. $f(x) = \frac{4}{x-4} + 2$

a) For what values of x is $f(x) = 0$?

b) Find P if A(p; o) lies on the graph.

c) Draw $f(x)$

d) What is the range of f.

e) Draw in the axis of symmetry of x and write down the equation.

10. Two painters can paint a room in one work shift of six hours.

a) How long will the work shift be if four painters complete the work at the same rate?

b) If the room needs to be painted in a work shift of 2,4 hours, how many painters are then needed if they work at the same rate.

c) Set up a table of values where x is the number of workers and y is the number of hours to complete the work.

d) Write down the equation that relates to y to x.

e) A two hour lunch break has to be included in any work shift. Write down the new equation.

f) A foreman who manages the team and does not paint, is also included in the formula. Write down the new equation.

g) How does the equation in (f) compare with the equation in (d)?

h) Draw the graph of (f).

i) Estimate from your graph (and show where you will read) how many hours, including lunch, are required if the work needs to be completed by a team of five people (including the foreman).

PROBABILITY

Lesson 39

Learning outcomes and assessment standards

Learning outcome 4: Data Handling and Probability
Assessment Standard
- Review the definition of probability
- Review set theory and venn-diagrams
- Use venn-diagrams to solve probability problems.
- Decide whether events are mutually exclusive.

Lesson

LC DVD

Definition of probability in words.

Probability of an event occurring = $\frac{\text{number of elements in the event}}{\text{number of possible elements}}$

Terminology you learnt in Grade 10

$n(\text{S})$ is the number of elements in the sample space and $\text{P(S)} = 1$

$n(\text{A})$ is the number of elements in event A

$\text{P(A)} = \frac{n(\text{A})}{n(\text{S})}$

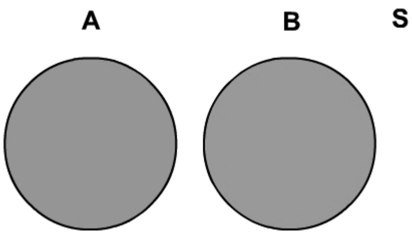
A′ means not A

$n(\text{A}')$ means the number of elements not in A

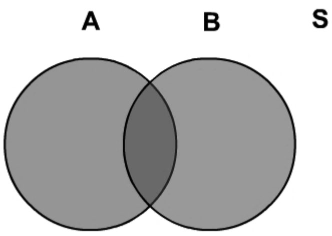
So $\text{P(A)} + \text{P(A)}' = \text{P(S)} = 1$

and $\text{P(A)}' = 1 - \text{P(A)}$

Venn-diagrams



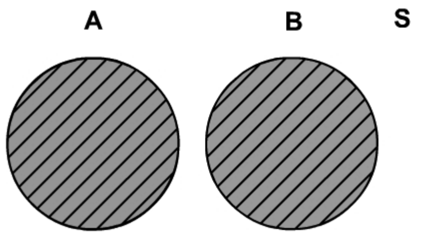
A and B are mutually exclusive events (there is no intersection)



A and B are not mutually exclusive events (they have certain elements in common).

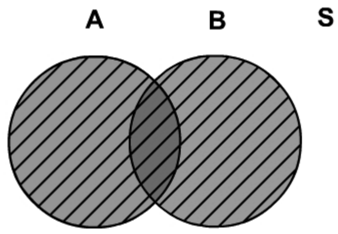
We sometimes shade in parts of a venn-diagram to illustrate union, intersection and complement.

NB: union means ‘or’
intersection means ‘and’
complement means ‘not’



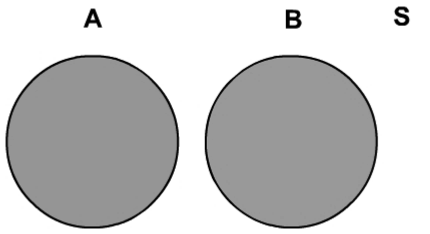
A∪B

(mutually exclusive)



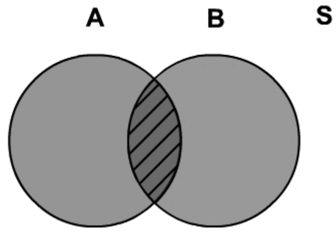
A∪B

(not mutually exclusive)



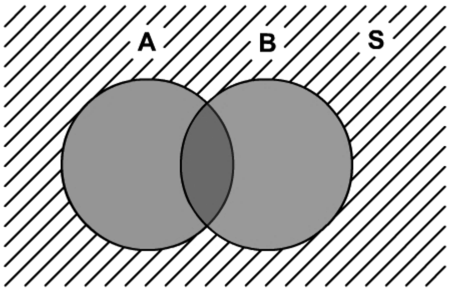
A∩B

(Nothing to shade)

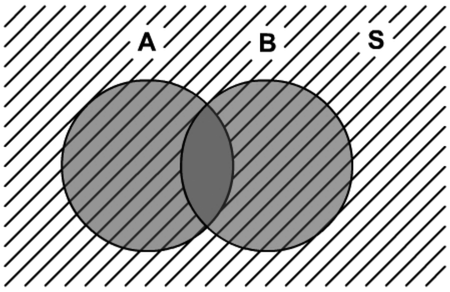


A∩B

(not mutually exclusive)



(A∪B)′

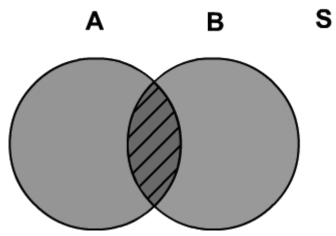


(A∪B)′

Example 1

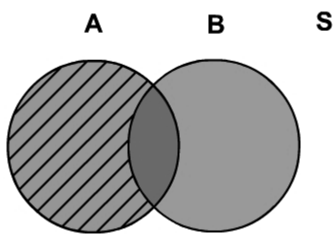
If A is the event that a person is arrested for drunken driving and B is the event that a person is a male, describe in words the shaded regions in each of the following diagrams.

a)

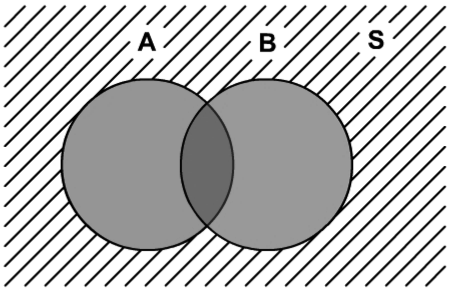


A male who is arrested for drunken driving

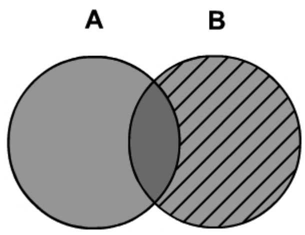
b) A female who is arrested for drunken driving.



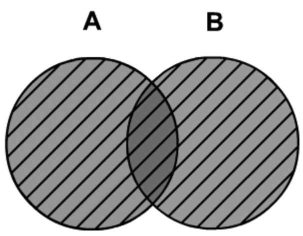
c) A female who is not arrested for drunken driving.



d) A male who is not arrested for drunken driving.



e) A male or a person arrested for drunken driving.



We will now use venn-diagrams to solve probability problems CV

1. There are 220 boys in Grade 11 at Sunnyvale High.

   160 play rugby

   110 play cricket

   70 play rugby and cricket

   Illustrate this information in a venn-diagram.

   Put the intersection in first.

   Find the probability that a boy chosen at random.

   a) Plays rugby or cricket.

   $\frac{200}{220} = \frac{10}{11}$

   b) Plays neither rugby nor cricket

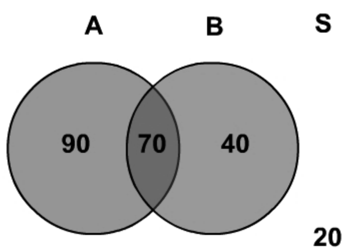
   $\frac{20}{220} = \frac{1}{11}$

   c) Plays only one of the two sports

   $\frac{130}{220} = \frac{13}{22}$

   d) Does not play both sports

   $\frac{150}{220} = \frac{15}{22}$



2. The probability that a Grade 9 student chooses maths as a subject is 0,7 and chooses geography is 0,3. The probability that he takes neither of the two is 0,2.

Find the probability that

a) He takes both maths and geography.

b) He takes only one of the two subjects –

(Hint: Always put the intersection in first – then build).

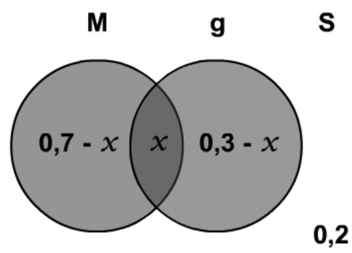
Solution

P(S) = 1 so

$0{,}7 - x + x + 0{,}3 - x + 0{,}2 = 1$

$0{,}7 + 0{,}3 - x + 0{,}2 = 1$

$x = 0{,}2$



a) 0,2

b) $0{,}5 + 0{,}1 = 0{,}6$

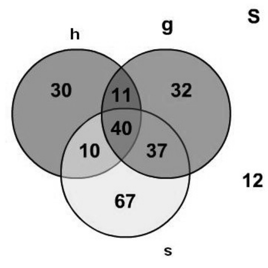
3. Three sets

A study into the other subjects taken by 239 maths students at Sherlock High School in 2006 yielded the following data.

- All but 12 wrote one or more of History, Geography or Science.
- 120 wrote Geography
- 154 wrote Science
- 51 wrote History and Geography
- 77 wrote Geography and Science
- 67 wrote Science but neither Geography nor History.
- 40 wrote all three.

Draw a diagram and answer the following questions.

Find the probability that a student selected at random takes



a) Neither Geography nor Science

$\frac{30 + 12}{239} = \frac{42}{239}$

b) Any two of the three subjects

$\frac{37 + 10 + 11}{239} = \frac{58}{239}$

c) At least 1 of the three subjects

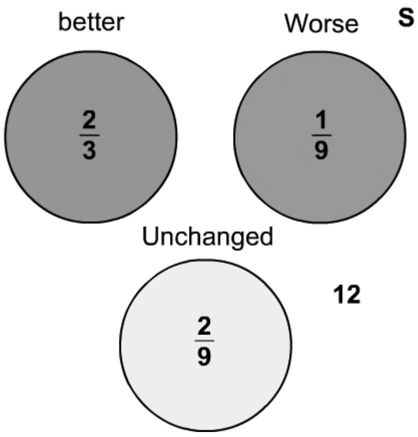
$1 - P(\text{none}) = 1 - \frac{12}{239} = \frac{227}{239}$

4. A doctor told a family anxious about the condition of their daughter that the odds that she would make a full recovery were 2:1.

The odds against her condition remaining unchanged were 7:2.

The odds against her condition becoming worse were 8:1

Determine whether the doctor's probabilities are possible.



$P(\text{Better}) = \frac{2}{3}$ $\qquad \frac{6}{9} + \frac{1}{9} + \frac{2}{9}$

$P(\text{Worse}) = \frac{1}{9}$ $\qquad = \frac{9}{9}$

$P(\text{unchanged}) = \frac{2}{9}$ $\qquad = 1$ The doctor is correct

Activities

INDIVIDUAL
PAIRS
FORMATIVE ASSESSMENT
SUMMATIVE ASSESSMENT

1. In an experiment, persons are asked to pick a letter from the first ten letters of the alphabet.

   A = {a; b; d; e; h} B = {a; e; j} C = {d; e; g; j}

   a) Draw a venn-diagram to represent the sample space and events A; B and C.

   b) Now find:

   (i) $P(A')$ (ii) $P(A \cap B)$ (iii) $P(A \cap B \cap C)$

   (iv) $P(A \cup C)$ (v) $P(A \cup B \cup C)'$ (vi) $P(A \cap B' \cap C)$

2. At Zibedele High School, there are 126 pupils in Form 4.

   44 pupils do additional mathematics

   112 pupils do mathematics and 90 pupils take science.

   All pupils who do additional mathematics also do mathematics.

   30 pupils do additional mathematics and science.

   80 pupils do mathematics and science.

   Draw a venn-diagram and find the probability that if a pupil is chosen at random he or she

   a) Takes neither of the three subjects.

   b) Only takes mathematics.

   c) Takes additional mathematics and not science.

   d) Takes mathematics and science but not additional mathematics.

   e) Takes mathematics or science.

3. The probability that a student will pass accounting is 0,7, the probability that he passes management is 0,8 and the probability that he passes both is 0,6. Determine the probability that

   a) He passes neither of the two

b) Only management c) Not accounting

d) Accounting or management (at least one of the two).

4. If $P(A) = 0,55$ and $P(B) = 0,4$ and $P(A \cap B) = 0,25$, find (by using a diagram)

a) $P(A')$ b) $P(A' \cap B)$ c) $P(B')$

d) $P(A \cup B)$ e) $P(A' \cup B)$ f) $P(A \cap B)'$

g) $P(A \cup B)'$

5. $P(A) = 0,48$ $P(B) = 0,5$ and $P(A \cup B) = 0,72$, find $P(A \cap B)$ (Use a diagram)

6. The probability that a man aged 60 will be alive in the year 2000 is $\frac{5}{8}$ and the probability that his wife will be alive is $\frac{5}{6}$. The probability that either one or both (at least one) are alive is $\frac{15}{16}$, find the probability that

a) Both will be alive b) Only the wife will be alive

c) Neither will be alive

7. Find the probability of drawing an ace or a heart from a pack of 52 cards.

8. A pupil attending university for the first time examined the previous year's statistics. He saw that 70% of the students passed mathematics, 30% passed chemistry and 80% passed at least one of the two subjects.

If he takes both these subjects, what will his chances be of passing

a) Both b) Just one subject

c) Neither.

9. The probability that a boy at High School chooses to play rugby is 0,5 and the probability that he chooses to play cricket is 0,3. The probability that he chooses to play neither of the two is 0,4.

Find the probability that

(i) He chooses to play both rugby and cricket.

(ii) He plays only rugby.

10. A study was done to determine the effect of 3 different drugs A, B and C in relieving headache pain. 80 patients were given the chance to use all 3 drugs. The following results were obtained:

40 reported relief from drug A.

35 reported relief from drug B.

40 reported relief from drug C.

21 reported relief from both drugs A and C.

18 reported relief from both drugs B and C

68 reported relief from at least one of the drugs.

7 reported relief from all three drugs

a) How many of the patients got relief from none of the drugs.

b) How many patients got relief from drugs A and B but not C.

c) What is the probability that a randomly chosen subject got relief from at least two of the drugs.

PROBABILITY 2

Lesson 40

Learning outcomes and assessment standards

Learning outcome 4: Data Handling and Probability
Assessment Standard

- Correctly identify dependent and independent events by using tables of diagrams.
- Use tree and venn-diagrams to solve probability problems (where events are not necessarily independent)

Overview

In this lesson you will:

- Recognise events that are independent or dependent.
- Decide when to draw a tree diagam to find the probability.
- Use $P(A \cap B) = P(A) \cdot P(B)$.
- Decide when to use a venn-diagram and when to use a tree diagram.

Lesson

DVD

Independent events

Events A and B are independent if the probability of A is not affected by the occurrence of B.

Examples

- A coin tossed twice
- A coin tossed and then a dice is thrown
- You guess two multiple choice answers
- A boy is born and then a girl
- Draw a card from a pack of 52, replace it and draw another card.

BUT

Dependent events

- Choose two cards from a pack of 52
- Choose two names out of a hat

$P(A \text{ and } B) = P(A \cap B)$

$= P(A) \times P(B)$

In statistics OR means add, AND means multiply.

Example

If a coin is tossed and then a dice thrown, what is the probability of getting a head and a six?

A: a head when the coin is tossed

$P(A) = \frac{1}{2}$

B: a six when the dice is thrown

$P(B) = \frac{1}{6}$

$P(A \text{ and } B)$

$= P(A \cap B)$

$= P(A) \times P(B)$

$= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Tree diagrams are useful

1. A box contains 3 coins.
   - one coin is fair
   - one coin has a head on both sides
   - one coin is weighted so that the probability of heads is $\frac{1}{3}$.

   If a coin is selected and tossed, find the probability that it is a head.

Solution

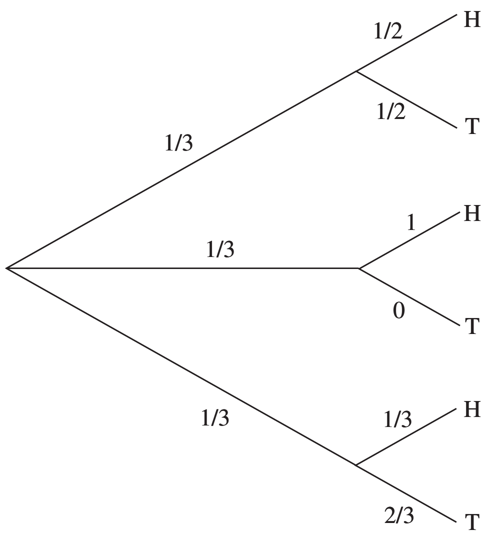
- Draw a tree
- First branch – select a coin

Follow the branches that end with a head.

$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

$= \frac{1}{6} + \frac{1}{3} + \frac{1}{9}$

$= \frac{11}{18}$



2. A bag contains 5 red beads and 3 blue beads. A second bag contains 6 blue beads and 4 red beads.

   If a bead is selected from one of the bags, find the probability the bead is blue.

- First select a bag
- Then select a bead

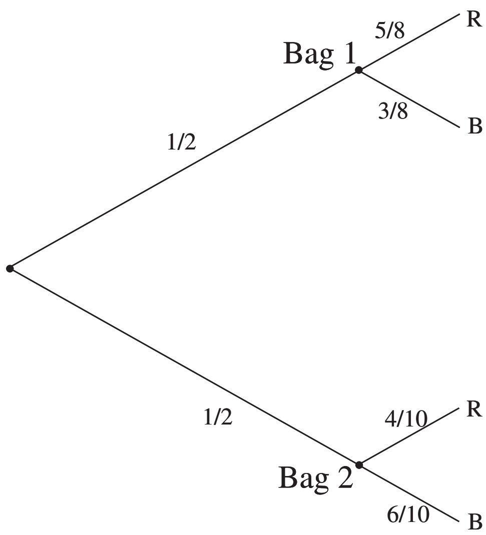
Solution

Follow the branches

$P(B) = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$

$= \frac{3}{16} + \frac{3}{10}$

$= \frac{39}{80}$



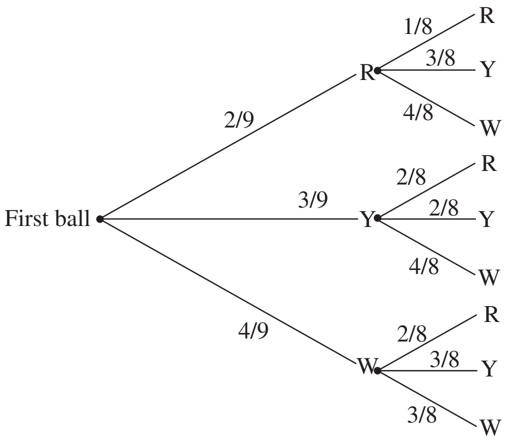
3. A bag contains 2 red balls, 3 yellow balls and 4 white balls. If two balls are selected, one after the other, find the probability

   a) That they are both the same colour.

   b) No red ball is selected.

Solution

Tree



a) Red and Red or Y and Y or W and W

$= \left(\frac{2}{9}\right)\left(\frac{1}{8}\right) + \left(\frac{3}{9}\right)\left(\frac{2}{8}\right) + \left(\frac{4}{9}\right)\left(\frac{3}{8}\right)$

$= \frac{2}{72} + \frac{6}{72} + \frac{12}{72}$

$= \frac{5}{18}$

b) Y and Y or Y and W or W and Y or W and W

$= \left(\frac{3}{9}\right)\left(\frac{2}{8}\right) + \left(\frac{3}{9}\right)\left(\frac{4}{8}\right) + \left(\frac{4}{9}\right)\left(\frac{3}{8}\right) + \left(\frac{4}{9}\right)\left(\frac{3}{8}\right)$

$= \frac{6}{72} + \frac{12}{72} + \frac{12}{72} + \frac{12}{72}$

$= \frac{7}{12}$

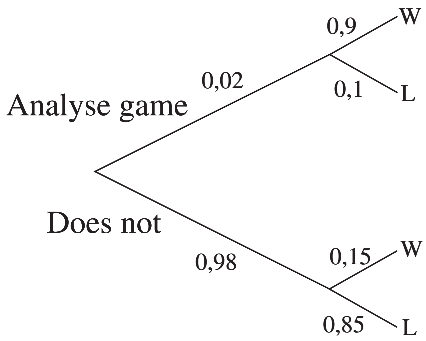
4. The probability of South Africa winning the Pro 20 cricket match when Smith analyses the game is 0,9. The probability of them winning the game when he does not analyse the game is 0,15. The probability of Smith analysing the game is 0,02. What is the probability that South Africa wins?

Solution

The tree tells a story so you need a number on each branch.

P(W) = (0,02)(0,9) + (0,98)(0,15)

= 0,165



DVD

PAIRS

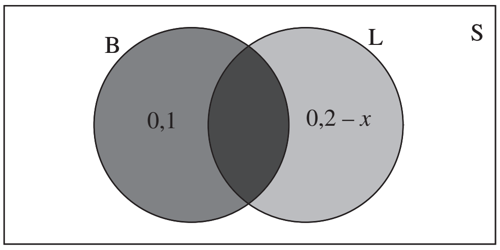
FORMATIVE ASSESSMENT

Activities

Venn or tree?

5. In a large group of people it is known that 10% have a hot breakfast, 20% have a hot lunch and 25% have a hot breakfast and hot lunch.
   Find the probability that a person chosen at random has a hot breakfast and lunch.

Solution:



There is an intersection.

Venn diagram.

Put the intersection in first.

$0{,}1 + 0{,}2 - x = 0{,}25$

$x = 0{,}3 - 0{,}25$

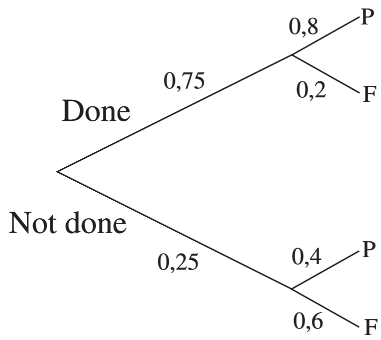
$x = 0{,}05$

6. A teacher discovers that only 75% of his students do their homework regularly. He calculated that if a student regularly did their homework, the probability of passing their exam is 0,8 and if the students did not do their homework the probability of passing is only 0,4. Calculate the probability that a randomly selected student passes the exam.

Solution

Tree – Events following each other.

$P(P) = (0{,}75)(0{,}8) + (0{,}25)(0{,}4)$
$= 0{,}7$



Activities

1. The probability that it will be sunny tomorrow is $\frac{1}{3}$.
   If it is sunny the probability that Jenny plays tennis is $\frac{4}{5}$. If it is not sunny, the probability that Jenny plays tennis is $\frac{2}{5}$. Find the probability that Jenny plays tennis.
2. A bag contains 5 red beads and 3 blue beads. A second bag contains 6 blue beads and 4 red beads. If a bag is chosen and then a bead selected, find the probability that the bead is blue.
3. It is estimated that $\frac{1}{4}$ of the drivers on the road after 11 pm have been drinking. If a driver has not been drinking, the probability that he will have an accident at that time is 0,004. If he has been drinking, the probability of an accident goes up to 0,02. What is the probability that a car selected at random at that time of night will have an accident?
   A policeman on the beat at 11:30 pm sees a car run into the lamp post.
   He jumps to the conclusion that the driver has been drinking. What is

the probability that he is correct? (Not really on the syllabus but quite challenging).

4. James can communicate with his friend overseas by three methods: letter, email or telephone. His friend does not want to write a letter so he communicates back by telephone or by email.

   Find the probability that

   a) James and his friend communicate using the same method.

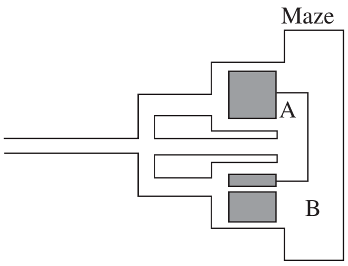
   b) One of them uses email.

5. A little old lady keeps her money in a sugar bowl. She has two R100 bills, four R50 bills, five R20 bills and nine R10 bills. The old lady reached into the bowl and grabs one bill and then another. What is the probability that

   a) She draws two R20 notes

   b) The total is less than R100?

6. A husband and wife team have reached the final round of a TV quiz show. To win the Grand Prize, the wife must choose one of two rooms and wait inside. The husband must walk through a simple maze, the paths of which lead to the two rooms. If the husband enters the room where his wife is waiting, they win the prize. The wife is shown a diagram of the maze. Which room should she choose?



(Work out P(A) and P(B)).

Lesson 41

FINANCIAL MATHEMATICS (1)

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.
Assessment Standard

- Use simple and compound decay formulae to solve problems, including straight line depreciation and depreciation on a reducing balance.

Overview

In this lesson you will:

- Revise Simple and Compound interest from Grade 10.
- Focus on the two types of Depreciation – Linear and Reducing Balance.

LC DVD

Lesson

PRIOR KNOWLEDGE

SIMPLE INTEREST

Consider an amount of R1 000 invested at 10% per annum **simple** interest:

Accumulated amount after one year:

$A_1 = 1\ 000 + 0{,}10 \times 1\ 000 = \text{R}1\ 100$ (Interest received is R100)

Accumulated amount after two years:

$A_2 = 1\ 100 + 0{,}10 \times 1\ 000 = \text{R}1\ 200$ (Interest received is R100)

Accumulated amount after three years:

$A_3 = 1\ 200 + 0{,}10 \times 1\ 000 = \text{R}1\ 100$ (Interest received is R100)

Accumulated amount after four years:

$A_4 = 1\ 300 + 0{,}10 \times 1\ 000 = \text{R}1\ 400$ (Interest received is R100)

Accumulated amount after five years:

$A_5 = 1\ 400 + 0{,}10 \times 1\ 000 = \text{R}1\ 500$ (Interest received is R100)

Note: The interest received each year is calculated using the original amount invested (R1000). This means that the interest received each year will always be the same (R100). The graph of this relationship will be a **linear function** (see diagram which follows).

COMPOUND INTEREST

Consider an amount of R1 000 invested at 10% per annum **compound** interest:

Accumulated amount after one year:

$A_1 = 1\ 000 + 0{,}10 \times 1\ 000 = \text{R}1\ 100$

Accumulated amount after two years:

$A_1 = 1\ 000 + 0{,}10 \times 1\ 000 = \text{R}1\ 100$

Accumulated amount after three years:

$A_3 = 1\ 210 + 0{,}10 \times 1\ 210 = \text{R}1\ 331$ (Interest received is R121)

Accumulated amount after four years:

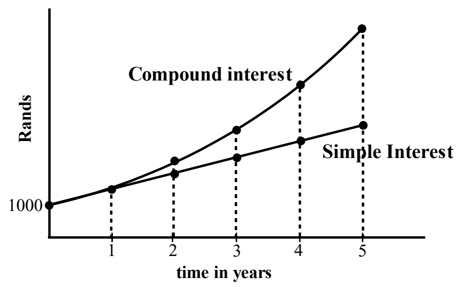
$A_4 = 1\ 331 + 0{,}10 \times 1\ 331 = R1\ 464{,}10$ (Interest received is R133,10)

Accumulated amount after five years:

$A_5 = 1\ 464{,}10 + 0{,}10 \times 1\ 464{,}10 = R1\ 610{,}51$ (Interest received is R146,41)

Note: The interest received at the end of the first year is the same as that received using the simple interest approach. However, at the end of the second year, the interest received is higher when using the compound interest approach because interest for the second year was calculated using the accumulated amount of the first year (R1 100). This will apply to each successive year and therefore the accumulated amount each year will increase exponentially. This is the case because both rates were 10 per annum. The graph of this relationship will therefore be an **exponential function** (see diagram which follows).

The graphical relationship between simple and compound Interest



Formulae for simple and compound interest

Simple interest

The formula which helps us to calculate the accumulated amount (future value) if an original amount is invested or loaned for *n* years at a rate of *r*% simple interest is given by:

$A = P(1 + in)$

where:

P = present value of the investment or loan (original amount at the beginning)

A = accumulated amount or future value of the investment or loan after *n* periods

n = time period in years

$i = \frac{r}{100}$ for the simple interest rate *r*% (*i* is in decimal form)

Compound interest

The formula which helps us to calculate the accumulated amount (future value) if an original amount is invested or loaned for *n* years at a rate of *r*% compound interest is given by:

$A = P(1 + i)^n$

where:

P = present value of the investment or loan (original amount at the beginning)

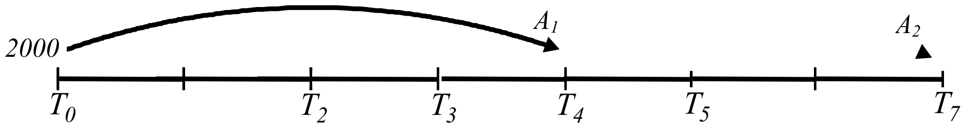
A = accumulated amount or future value of the investment or loan after *n* periods

n = time period

$i = \frac{r}{100}$ for the simple interest rate *r*% (*i* is in decimal form)

Example

An amount of R2 000 is invested for 4 years at 12% per annum simple interest. Thereafter, the accumulated amount is invested for another 3 years at 11% per annum compound interest. Calculate how much money will have been saved at the end of the seven year period.



For the first four years:

$A_1 = P(1 + in)$

$\therefore\ A_1 = 2\ 000(1 + 0{,}12 \times 4)$

$\therefore\ A_1 = R2\ 960$

For the remaining three years:

$A_2 = P(1 + i)^n$ where $P = A_1$

$\therefore\ A_2 = 2\ 960(1 + 0{,}11)^3$

$\therefore\ A_2 = R4\ 048{,}19$

Note:

You could have done the calculation in one line as follows:

$\therefore\ A = 2\ 000(1 + 0{,}12 \times 4)\ .\ (1 + 0{,}11)3$

$\therefore\ A = R4\ 048{,}19$

Link to Activity 1

Depreciation

When equipment loses value over time, we say that the equipment is depreciating in value. For example, it is generally the case that the moment a new motor car is driven out of the shop, its value depreciates substantially. Obviously, due to wear and tear, the car will lose its value over time.

Book value: is the value of equipment at a particular time after depreciation has been taken into account.

Scrap value: is the book value of the equipment at the end of its useful life.

There are two types of depreciation:

Linear depreciation and reducing balance depreciation

Linear depreciation (straight-line depreciation)

With straight-line depreciation, equipment is depreciated by a percentage of its original value. It works in the same way as simple interest, but the value **decreases** rather than increases as with simple interest.

Consider, for example, the value of a car costing R200 000 after 6 years if it depreciates at a rate of 10% per annum using linear depreciation.

$A_1 = 200\ 000 - 0{,}10 \times 200\ 000$

$\therefore\ A_1 = R180\ 000$ (The car's value depreciated by R20 000)

$A_2 = 180\ 000 - 0{,}10 \times 200\ 000$

$\therefore\ A_2 = R160\ 000$ (The car's value depreciated by a further R20 000)

$A_3 = 160\ 000 - 0{,}10 \times 200\ 000$

$\therefore\ A_3 = R140\ 000$ (The car's value depreciated by a further R20 000)

$A_4 = 140\ 000 - 0{,}10 \times 200\ 000$

$\therefore\ A_4 = R120\ 000$ (The car's value depreciated by a further R20 000)

$A_5 = 120\ 000 - 0{,}10 \times 200\ 000$

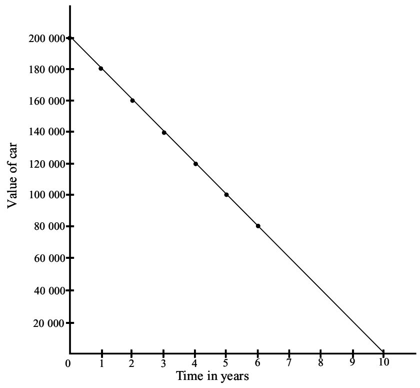
$\therefore\ A_5 = R100\ 000$ (The car's value depreciated by a further R20 000)

$A_6 = 100\ 000 - 0{,}10 \times 200\ 000$

$\therefore\ A_6 = R80\ 000$ (The car's value depreciated by a further R20 000)

Note: The car's value depreciates by the same amount of R20 000 each year. After 6 years, the car is worth R80 000.

We can represent this information on a graph as follows:



Note: The graph of this depreciation takes on the shape/trend of a ***linear function*** (straight-line graph). The value of the car will eventually be zero, because the graph will cut the horizontal axis where the value of the car as read from the vertical axis will be zero.

A useful formula to calculate linear depreciation is:

Consider, for example, the value of a car costing R200 000 after 6 years if it depreciates at a rate of 10% per annum using linear depreciation.

$A = P(1 - in)$

$\therefore\ A = 200\ 000(1 - 0{,}10 \times 6)$

$\therefore\ A = R80\ 000$

Reducing-balance depreciation

With reducing-balance depreciation, equipment is depreciated by a percentage of its previous value. It works in the same way as compound interest, but the value **decreases** rather than increases as with compound interest.

Consider, for example, the value of the car costing R200 000 after 4 years if it depreciates at a rate of 10% per annum using reducing balance depreciation.

$A_1 = 200\ 000 - 0{,}10 \times 200\ 000$

$\therefore\ A_1 = R180\ 000$ (The car's value depreciated by R20 000)

$A_2 = 180\ 000 - 0{,}10 \times 180\ 000$

$\therefore\ A_2 = \text{R}162\ 000$ (The car's value depreciated by a further R18 000)

$A_3 = 162\ 000 - 0{,}10 \times 162\ 000$

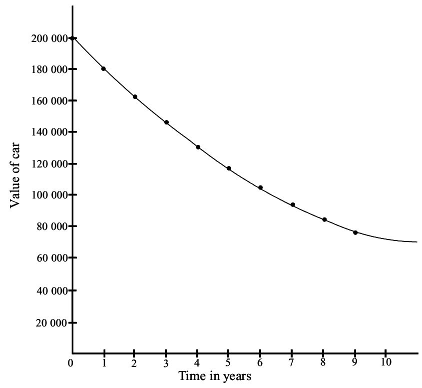
$\therefore\ A_3 = \text{R}145\ 800$ (The car's value depreciated by a further R16 200)

$A_4 = 145\ 800 - 0{,}10 \times 145\ 800$

$\therefore\ A_4 = \text{R}131\ 220$ (The car's value depreciated by a further R14 580)

Note: The car's value depreciates by different lesser amounts each year. After 4 years, the car is worth R131 220. Depreciation is calculated on the reducing balance each year.

We can represent this information on a graph as follows:



Note: The graph of this depreciation takes on the same shape/trend as a decreasing ***exponential function***. The car will always have some value as the years progress. This is because the exponential graph never cuts the horizontal axis so as to produce a value of zero rands (as read off the vertical axis).

Link to Activity 2

A useful formula to calculate reducing balance depreciation is:

Consider, for example, the value of the car costing R200 000 after 4 years if it depreciates at a rate of 10% per annum using reducing balance depreciation.

$A = P(1 - i)^n$

$\therefore\ A = 200\ 000(1 - 0{,}10)^4$

$\therefore\ A = \text{R}131\ 220$

Example 1

What will the book value of a car be after 5 years if the rate of depreciation is 16,1% p.a. and the car's original purchase price was R72 000 where depreciation is based on:

(a) the straight-line method?

(b) the reducing-balance method?

Solution

(a) $A = 72\ 000(1 - 0{,}161 \times 5)$

$\therefore\ A = \text{R}14\ 040$

(b) $A = 72\ 000(1 - 0{,}161)^5$

$\therefore\ A = \text{R}29\ 932{,}45$

Example 2

Calculate the original price of a laptop computer if its depreciated value after 6 years is R1 200 and the rate of depreciation was 14% per annum calculated using:

(a) the straight-line method?

(b) the reducing-balance method?

Solution

(a) $A = P(1 - i)^n$

$\therefore\ 1\,200 = P(1 - 0{,}14 \times 6)$

$\therefore\ 1\,200 = P(0{,}16)$

$\therefore\ \frac{1\,200}{0{,}16} = P$

$\therefore\ P = R7\,500$

(b) $A - P(1 - in)$

$\therefore\ 1\,200 = P(1 - 0{,}14)^6$

$\therefore\ 1\,200 = P(0{,}86)^6$

$\therefore\ \frac{1\,200}{(0{,}86)^6} = P$

$\therefore\ P = R2\,966{,}13$

Example 3

A car costs R69 000 and, after 8 years, has a scrap value of R7000. Find the annual depreciation rate if it is calculated using:

(a) the straight line method.

(b) the reducing balance method.

Solution

(a) $7\,000 = 69\,000(1 - 8i)$

$\therefore\ 7\,000 = 69\,000 - 552\,000i$

$\therefore\ 552\,000i = 69\,000 - 7\,000$

$\therefore\ 552\,000i = 62\,000$

$\therefore\ i = \frac{62\,000}{552\,000}$

$\therefore\ i = 0{,}1123188406$

$\therefore\ r = 11{,}2\%$ per annum

(b) $7\,000 = 69\,000(1 - i)^8$

$\therefore \frac{7\,000}{69\,000} = (1 - i)^8$

$\therefore \left(\frac{7\,000}{69\,000}\right)^{\frac{1}{8}} = 1 - i$

$\therefore\ i = 1 - \left(\frac{7\,000}{69\,000}\right)^{\frac{1}{8}}$

$\therefore\ i = 0{,}248755823$

$\therefore\ r = 24{,}9\%$ per annum

Link to Activity 3

Summary

In this lesson, we looked at the differences between simple and compound interest. Clearly, there is more money to be made if you invest your money using the compound interest option.

We then discussed linear and reducing balance depreciation, showing how things can lose value over time.

In the next lesson, we will focus on different compounding periods as well as nominal and effective interest rates.

Activity 1

1. Find the future value of R4 000 invested for 5 years at

   (a) 14% p.a. simple interest.

   (b) 14% p.a. compound interest.

2. Find the present value of an amount which accumulated to R13 000 in 6 years if the interest was

(a) 12% p.a. simple interest.

(b) 12% p.a. compound interest.

3. R70 000 is invested at 9% p.a. simple interest for 3 years. Thereafter, the total amount is reinvested in a different financial institution at 8% p.a. compound interest for 2 more years. What is the future value of the investment after the five-year period?

Activity 2

Use the graph to answer the following questions:

(a) Determine the value of the car after 7 years: ____________________

(b) Determine the value of the car after 9 years: ____________________

(c) How long did the car take to depreciate to R60 000? _____________

(d) How long did the car take to depreciate to R40 000? _____________

(e) What will the value of the car be after 10 years? ________________

Activity 3

1. A car is purchased for R250 000. The car depreciates at a rate of 12% per annum. What is the car worth after 5 years if depreciation is calculated using:

   (a) the straight-line method.

   (b) the reducing-balance method.

2. Calculate the original price of a laptop computer if its depreciated value after 7 years is R3 200 and the rate of depreciation was 12% per annum calculated using:

   (a) the straight-line method?

   (b) the reducing-balance method?

3. A photocopying machine costs R140 000 and has a scrap value of R19 000 after 10 years. Find the annual rate of depreciation if it is calculated using:

   (a) the straight-line method.

   (b) the reducing-balance method.

4. A motor vehicle currently has a book value of R56 000. The rate of depreciation was 14% per annum using the reducing balance method. Calculate the original price of the motor vehicle, if it was bought 5 years ago.

5. The computers for a business are currently worth R400 000. Calculate the value of these computers after 6 years, if the rate of depreciation is 16% per annum calculated on a linear basis.

6. A school buys a photocopying machine for R750 000. Calculate the scrap value of the machine after 6 years if the rate of depreciation is 13% per annum calculated on the reducing balance scale.

FINANCIAL MATHEMATICS (2)

Lesson 42

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.
Assessment Standard
Demonstrate an understanding of different periods of compounding growth and decay (including effective compounding growth and decay and including effective and nominal interest rates.

Overview

In this lesson you will:

- Learn about different compounding periods.
- Learn about nominal and effective interest rates.

Lesson

DIFFERENT COMPOUNDING PERIODS

Compound interest is usually quoted as an annual rate. However, it can happen that interest is calculated over shorter time periods during a year.

Interest can be calculated:

⇨ **annually**: once per year (usually at the end of the year)

⇨ **half-yearly(semi-annually)**: twice per year (every six months)

⇨ **quarterly**: four times per year (every three months)

⇨ **monthly**: twelve times per year (every one month)

⇨ **daily**: 365 times per year (excluding leap years)

Example 1

Calculate the future value of an investment of R10 000 after three years at an interest rate of 18% per annum compounded:

(a) annually

(b) half-yearly

(c) quarterly

(d) monthly

(e) daily (excluding leap years)

(a) Annually

$A = 10\,000(1 + 0{,}18)^3$

$\therefore\ A = 10\,000(1{,}18)^3$

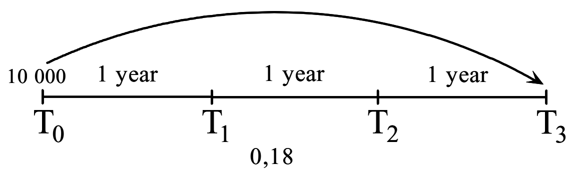
$\therefore\ A = R16\,430{,}32$

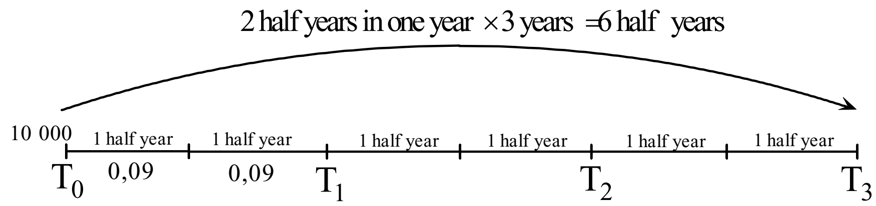
(b) Semi-annually

$A = 10\,000\left(1 + \frac{0{,}18}{2}\right)^6$

$\therefore\ A = 10\,000(1{,}09)^6$

$\therefore\ A = R16\,771$





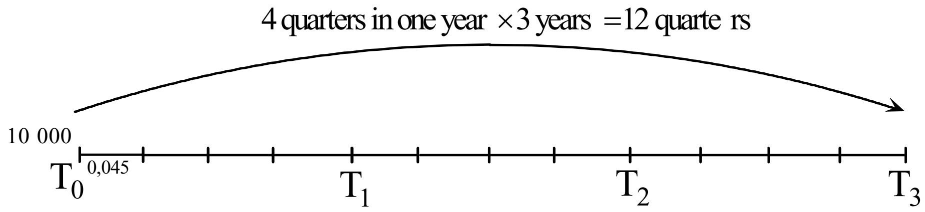
$$\frac{0{,}18}{2} = 0{,}09 \text{ interest rate per half year}$$

(c) Quarterly

$A = 10\ 000\left(1 + \frac{0{,}18}{4}\right)^{12}$

$\therefore\ A = 10\ 000(1{,}045)^{12}$

$\therefore\ A = R16\ 958{,}14$



$$\frac{0{,}18}{4} = 0{,}045 \text{ interest rate per quarter}$$

(d) Monthly

$A = 10\ 000\left(1 + \frac{0{,}18}{12}\right)^{36}$

$\therefore\ A = 10\ 000(1{,}015)^{36}$

$\therefore\ A = R17\ 091{,}40$

(e) There are 365 days in a year (excluding leap years).

Therefore, there are 365 × 3 = 1095 days in three years.

The daily interest rate will be

$\frac{0{,}18}{365} = 0{,}00049315$

$A = 10\ 000\left(1 + \frac{0{,}18}{365}\right)^{1095}$

$\therefore\ A = R17\ 157{,}78$

Note: As the number of compounding periods increase (for the same rate) during a year, the greater the accumulated amount at the end.

Link to Activity 1

Nominal and effective interest rates

Nominal interest rates

A **nominal rate** is quoted as an annual rate, without taking into consideration the effect of compounding periods, which are shorter than the annual period. For example, 18% per annum compounded monthly is a nominal rate. The quoted annual rate is 18% but the fact that the interest is compounded monthly means that the accumulated amount at the end will be higher. The quoted annual rate of 18%

(no compounding taken into account) will yield a lower accumulated amount than 18% compounded monthly.

We will refer to per annum as p.a.
compounded quarterly c.q.
compounded monthly c.m.
compounded daily c.d.

Consider the following example to illustrate this.

R2 000 invested for one year at 18% per annum without monthly compounding:

$A = 2000(1 + 0{,}18)^1$

$\therefore\ A = R2\ 360$

R2 000 invested for one year at 18% per annum with monthly compounding:

$A = 2000\left(1 + \frac{0{,}18}{12}\right)^{12}$

$\therefore\ A = R2391{,}24$

Nominal rates:
r = 5% p.a. c.m/differ
OR
r = 18% p.a. c.q./differ

Clearly, the monthly compounding yielded a higher accumulated amount than the quoted annual rate of 18% without compounding.

It is possible to determine an annual rate that will yield the same accumulated amount as the nominal rate which was compounded monthly.

Method 1

$A_{nom} = A_{eff}$

$\therefore\ P\,(1 + i_1)^{n1} = P\,(1 + c_2)^{n2}$

$\therefore\ (1 + i_1)^{n1} = P\,(1 + i_2)^{n2}$

$(1 + \frac{0{,}18}{12})^{12} = (1 + i)$

$\therefore\ i = 0{,}195618171$

$\therefore\ r = 19{,}56\%$ per annum compounded

Method 2

The amount invested was R2 000. This accumulated to R2 391,24 by monthly compounding. So the interest gained iss R391,24.

Now: $\frac{391{,}24}{2000} \times 100 = 19{,}562$

$\therefore\ r = 19{,}56\%$ p.a. c.a.

This means that a nominal rate of 18% per annum compounded monthly (p.a.c.m.) is the same as the effective rate per annum of 19,56% p.a.c.a.

Effective interest rates

As discussed previously, effective annual interest rates are equivalent annual rates that yield the same accumulated amount as rates with different compounding periods (monthly, quarterly, half yearly, daily). Effective rates are higher than quoted nominal rates. We get two types of effective rates. They are effective per period or effective annual.

For effective annual rates: (i_{eff})

$2000(1 + 0{,}19562)^1 = 2000\left(1 + \frac{0{,}18}{12}\right)^{12}$ (both equal R2 391,24)

$\therefore\ 2000(1 + i_{\text{eff}}) = 2000\left(1 + \frac{0{,}18}{12}\right)^{12}$

$\therefore\ 1 + i_{\text{eff}} = \left(1 + \frac{0{,}18}{12}\right)^{12}$

We can therefore create a formula that helps to calculate the effective rate when given a nominal rate or vice versa. The formula can be written as follows:

$$\boxed{1 + i_{\text{eff}} = \left(1 + \frac{i_{\text{nom}}}{n}\right)^{n}}$$

where:

i_{eff} = effective rate (annual)

i_{nom} = nominal rate per period

n = number of compoundings per year

We have not yet established how to compare interest rates offered by two financial institutions. For example, if Bank A offers you a rate of 14% per annum compounded monthly and Bank B offers a rate of 15% per annum compounded semi-anually, which offer should we accept? The only way to really make such a choice is to "translate" both these rates into rates that read the same, that is per annum compounded annually, or per month compounded monthly. That is rate where the **stated period and the compounding periods are the same**. These rates we refer to as **effective rates**. We cannot compare them otherwise.

If an institution charges you 12% per annum, compounded monthly, we say that this is a **nominal monthly rate**, since the per annum (stated period) and the compounded monthly (compounded period) **differ**. Now if we pay 12% per annum, and the compounding took place monthly, then we will compound interest 12 times a year, since a year has twelve months. As simple as that. Thus we will pay $r = \frac{12}{12}\% = 1\%$ per month compounded monthly. This rate is referred to as an **effective rate per period**.

Note that the per annum changed to per month. The same for 12% p.a. compounded semi-annually translates into $\frac{12}{2}\% = 6\%$ per semi-annum compounded semi-annually and 12% p.a. compounded quarterly translates into $\frac{12}{4}\% = 3\%$ per quarter compounded quarterly. All of these are effective rates per period.

The problem arises when we look at translating the compounding period to a different period, such as compounded monthly to compounded weekly or compounded daily or quarterly etc. This will involve a little bit of effort, not just a simple division exercise. Remember we are not changing the rate, we merely translate it into what we require to know. The information will still have the same effect in our problems.

When we want to translate x% p.a. compounded monthly (the nominal monthly rate) to y% per semi-annum compounded semi-annually (the effective semi annual rate), or even to z%p.a. compounded semi-annually (the nominal semi-annual rate), we need to do so that both rate reflect the equivalence required. Let us try.

We wish to translate x% p.a. compounded monthly to a percentage per annum compounded semi-annually.

Since we do not work with percentages, $i_{12} = \frac{x}{100}$ and $i_{2} = \frac{y}{100}$. If in one year we compound monthly. we then calculated interest 12 times that year, and semi-annual compounding will result into interest calculated twice that year. Since these rates are equivalent, and we invest P into each account, at the end of the day, each future value F will be the same.

Thus:

$F_v = P_v\,(1 + \frac{i_{12}}{12})^{12}$ $= P_v\,(1 + \frac{i_2}{2})^2$

we must work with the effective monthly rate and the period becomes the amount of times we compounded in one year

we must work with the effective semi-ann. rate and the period becomes the amount of times we compounded in one year

In general, the translation of the compound formula:

An amount at compound interest:

$F_v = P_v(1 + i)^n$

Annually:	Semi-annually:	Quarterly:	Where:
$F_v = P_v(1 + i)^n$	$F_v = P_v(1 + \frac{i_2}{2})^{2n}$	$F_v = P_v(1 + \frac{i_4}{4})^{4n}$	$i = \frac{r}{100}$
Monthly:	Daily:		n = years
$F_v = P_v(1 + \frac{i_{12}}{12})^{12n}$	$F_v = P_v(1 + \frac{i_{365}}{365})^{365n}$		P_v = Present value
			F_v = Future value

Example 2

(a) Convert a nominal rate of 18% per annum compounded monthly to an effective rate per annum.

$1 + \mathrm{i_{eff}} = \left(1 + \frac{\mathrm{i_{nom}}}{\mathrm{n}}\right)^{\mathrm{n}}$

$\therefore\ 1 + \mathrm{i_{eff}} = \left(1 + \frac{0{,}18}{12}\right)^{12}$

$\therefore\ \mathrm{i_{eff}} = \left(1 + \frac{0{,}18}{12}\right)^{12} - 1$

$\therefore\ \mathrm{i_{eff}} = (1{,}015)^{12} - 1$

$\therefore\ \mathrm{i_{eff}} = 0{,}195618171$

$\therefore\ \mathrm{i_{eff}} = 19{,}6\%$

(b) Convert an effective rate of 12,5% per annum, to a nominal rate per annum compounded half yearly.

$1 + \mathrm{i_{eff}} = \left(1 + \frac{\mathrm{i_{nom}}}{\mathrm{n}}\right)^{\mathrm{n}}$

$\therefore\ 1 + 0{,}125 = \left(1 + \frac{\mathrm{i_{nom}}}{2}\right)^{2}$

$\therefore\ 1{,}125 = \left(1 + \frac{\mathrm{i_{nom}}}{2}\right)^{2}$

$\therefore\ (1{,}125)^{\frac{1}{2}} = 1 + \frac{\mathrm{i_{nom}}}{2}$

$\therefore\ (1{,}125)^{\frac{1}{2}} = 1 = \frac{\mathrm{i_{nom}}}{2}$

$\therefore\ 0{,}060660171 = \frac{\mathrm{i_{nom}}}{2}$

$\therefore\ 0{,}121320343 = \mathrm{i_{nom}}$

$\therefore\ \mathrm{r} = 12{,}1\%$ per annum compounded half yearly

Example 3

R13 000 is invested for 3 years at 15% per annum compounded monthly.

(a) Calculate the future value of the investment using the nominal rate.

$A = 13\,000\left(1 + \frac{0,15}{12}\right)^{36}$

$\therefore\ A = 13\,000(1,0125)^{36}$

$\therefore\ A = R20\,331,27$

(b) Convert the nominal rate of 15% per annum compounded monthly to the equivalent effective rate (annual).

$1 + i_{eff} = \left(1 + \frac{0,15}{12}\right)^{12}$

$\therefore\ i_{eff} = (1,0125)^{12} - 1$

$\therefore\ i_{eff} = 0,160754517$

(c) Now use the annual effective rate to show that the same accumulated amount will be obtained as when using the monthly rate.

$A = 13\,000(1 + 0,160754517)^{3}$

$A = 13\,000(1,160754517)^{3}$

$A = R20\,331,27$

Summary

In this lesson we demonstrated the relationship between nominal and effective rates. Nominal rates are quoted annual rates but don't take into consideration the effects of different compounding periods. The effective rate is the annual rate that yields the same accumulated amount as the calculation involving different compounding periods.

In the next Lesson, we will learn about fluctuating interest rates, withdrawals and additional payments.

Link to Activity 2

Activity 1

1. Calculate the future value of an investment of R20 000 after 6 years, if the interest rate is 15% per annum compounded:

   (a) annually

   (b) half yearly

   (c) quarterly

   (d) monthly

   (e) daily (excluding leap years)

2. How much more money would you have over a period of 15 years on a savings of R60 000 if the interest rate of 12% per annum was compounded monthly, as compared to an interest rate of 12% compounded annually?

3. R30 000 is invested for 4 years at an interest rate of 16% per annum compounded quarterly. Thereafter, the accumulated amount is reinvested for a further 5 years at an interest rate of 15% per annum compounded

semi-annually (half yearly). Calculate the value of the investment at the end of the 9 year period.

4. Determine which of the following two savings options is a better investment over a period of one year, if the interest is calculated at:

   A. 14% per annum compounded monthly

   B. 16% per annum compounded quarterly

5. An amount of money was invested 6 years ago and it is now, after 6 years, worth R1 200 000. The interest rate for the savings period was 18% per compounded monthly. What was the amount that was originally invested 6 years ago?

Activity 2

1. Convert the following nominal rates to equivalent effective rates:

   (a) 14% per annum compounded half yearly.

   (b) 16% per annum compounded quarterly.

   (c) 12% per annum compounded monthly.

   (d) 10% per annum compounded daily.

2 (a) Convert an effective rate of 14,5% per annum, to a nominal rate per annum compounded half yearly.

   (b) Convert an effective rate of 13,2% per annum, to a nominal rate per annum compounded quarterly.

   (c) Convert an effective rate of 10,5% per annum, to a nominal rate per annum compounded monthly.

3. A man invests R24 000 at 16% per annum compounded quarterly for a period of 12 years.

   (a) Calculate the future value of the investment using the nominal rate.

   (b) Convert the nominal rate of 16% per annum compounded quarterly to the equivalent effective rate (annual).

   (c) Now use the annual effective rate to show that the same accumulated amount will be obtained as when using the monthly rate.

4. Mpho deposited R500 000 into a fixed deposit savings account for a period of six years. The accumulated amount at the end of the six year period is R650 000. Calculate the interest rate paid in each of the following cases:

   (a) The annual effective rate.

   (b) The nominal rate per annum compounded monthly.

Lesson 43

FINANCIAL MATHEMATICS (3)

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.
Assessment Standard
Demonstrate an understanding of different periods of compounding growth and decay.

Overview

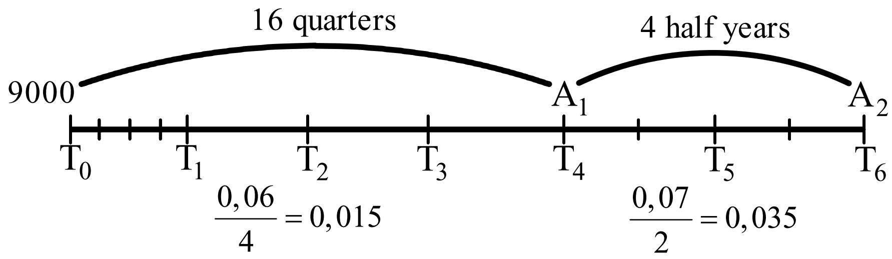
In this lesson you will learn about:

- Future and Present Value calculations involving changing Interest Rates.

Lesson

Example 1

An investment of R9 000 earns 6% per annum compounded quarterly for a period of four years. Thereafter, the interest rate changes to 7% per annum compounded semi-annually for a further two years. Calculate the future value of the investment at the end of the six-year period.



In order to calculate the future value of the investment at $\mathbf{T}_6$, we need to grow R9 000 through two interest rates. To do this, we first grow the money to $\mathbf{T}_4$ through the 6% interest rate to obtain $\mathbf{A}_1$. We then grow this amount further through the 7% interest rate in order to obtain $\mathbf{A}_2$, the future value after 6 years.

Method 1 (Long way)

$A_1 = 9\ 000\left(1 + \frac{0,06}{4}\right)^{16} = 11420,\ 86993$ (At T_4)

$\therefore\ A_2 = 11420,\ 86993.\left(\frac{1 + 0,07}{2}\right)^4$

$\therefore\ A_2 = \text{R}13\ 105,71$

(At T_6)

Method 2 (Recommended)

$A_1 = 9\ 000\left(1 + \frac{0,06}{4}\right)^{16} \cdot\left(\frac{1 + 0,07}{2}\right)^4$

$\therefore\ A_2 = 9\ 000(1,015)^{16} \cdot 1,035^4$

$\therefore\ A_2 = \text{R}13\ 105,71$

Link to Activity 1

A FORMULA FOR CALCULATING THE PRESENT VALUE (P)

Consider the formula $A = P(1 + i)^n$.

This formula can be rearranged as follows:

$A = P(1 + i)^n$

$\therefore \frac{A}{(1 + i)^n} = P$

$\therefore A(1 + i)^{-n} = P$ (use the exponent definition $\frac{1}{a^n} = a^{-n}$)

$\therefore P = A(1 + i)^{-n}$

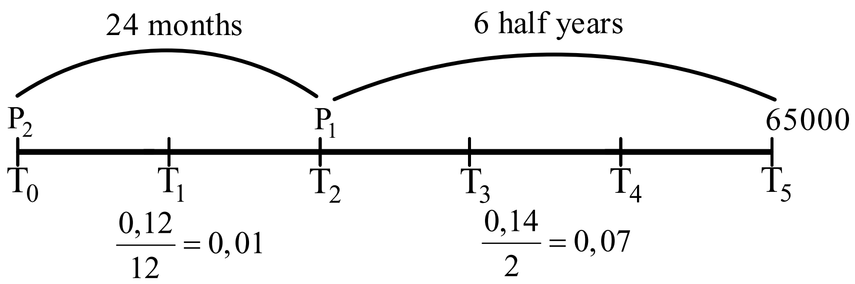
Therefore, we now have two formulae available to us:

$A = P(1 + i)^n$ (to find A given P)
$P = A(1 + i)^{-n}$(to find P given A)

In the next example, we focus on a quicker way of doing the calculation, involving the formula for P.

Example 2

Peter invests a certain sum of money for 5 years at 12% per annum compounded monthly for the first two years and 14% per annum compounded semi-annually for the remaining term. The money grows to R65 000 at the end of the 5-year period. How much did Peter originally invest?



(Using: $P = A\ (1 + i)^{-n}$ because we are moving backwards on the time line)

$x = 65\ 000(1{,}07)^{-6} \times (1{,}01)^{-24}$

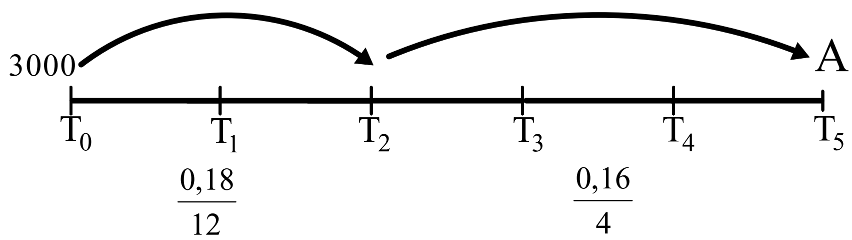
$x = \text{R}34\ 111{,}\ 26$

Summary

The following example will clearly illustrate the use of the two formulae.

- Using the formula $A = P(1 + i)^n$

Suppose that R3 000 is invested for 5 years. The interest rate for the first two years is 18% per annum compounded monthly. For the remaining three years, the interest rate changes to 16% per annum compounded quarterly. Calculate the value of the investment at the end of the five year period.

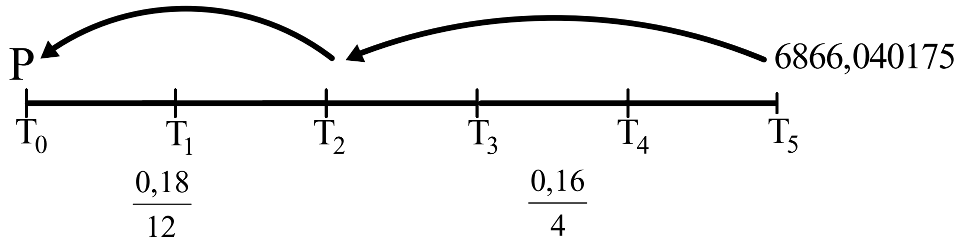


$$A = 3000\left(1 + \frac{0,18}{12}\right)^{24}\left(1 + \frac{0,16}{4}\right)^{12} = 6866,040175$$

Note: The exponents are **positive** and the movement on the time-line is from **left to right** because interest on the R3000 is growing.

- Using the formula $P = A(1 + i)^{-n}$

Suppose that R6866,040175 is received if a certain amount of money was invested 5 years ago. The interest rate for the first two years was 18% per annum compounded monthly. For the remaining three years, the interest rate changed to 16% per annum compounded quarterly. What was the original amount invested?



$$P = 6866,040175\left(1 + \frac{0,16}{4}\right)^{-12}\left(1 + \frac{0,18}{12}\right)^{-24} = R\ 3000$$

Notice: The exponents are <u>negative</u> and the movement on the time-line is from <u>right to left</u> because interest on the R6866,040175 is being removed.

Link to Activity 2

Activity 1

1. R5000 is deposited into a savings account. The interest rate for the first four years is 7% per annum compounded quarterly. Thereafter, the interest rate changes to 8% per annum compounded semi-annually. Calculate the value of the investment at the end of the tenth year.

2. Mark invests R2000 for a period of seven years. During the first four years, the interest rate is 18% p.a. compounded monthly. Thereafter, interest changes to 24% p.a. compounded semi-annually. Calculate the future value of the investment after seven years.

3. Simphiwe deposits R3000 into a savings account paying 13% per annum compounded monthly. After five years, the interest rate increases by 1%. Three years later, the interest rate decreases by 2%. Calculate the value of her investment after ten years.

4. Mvelo invests R6000 into an account for a period of 12 years. The interest rate for the first seven years is 8% per annum compounded monthly. For the next five years, the interest rate changes to 10% per annum compounded half-yearly.

   (a) Convert the nominal rates to annual effective rates.

   (b) Use the effective rates and calculate the future value of the savings at the end of the 12-year period.

Activity 2

1. Malibongwe deposits a certain amount in a savings account. It grows to an amount of R13 000 after seven years. The interest rate during the first four years is 9% per annum compounded annually and for the remaining three years is 12% per annum compounded monthly. How much is this amount?
2. Simone wants a sum of R10 000 000 in eight years from now. How much must she invest now if interest is 15% p.a. compounded monthly for the first six years, and 20% per annum compounded quarterly for the remaining two years?
3. Mark wants to save for an overseas trip in three years' time. He will need an amount of R50 000 for the trip. The interest rate during the first year is 14% per annum compounded quarterly. For the remaining two years, the interest rate is 11% per annum compounded monthly. What must Mark invest now in order to receive R50 000 in three years' time?
4. Justine receives a certain amount of money as a birthday gift. She wants to invest this money in a saving account in order to buy a motor car when she matriculates four years from now. The expected cost of the motor car in four years' time is R100 000. The interest rate during the first two years of the savings period is 14% per annum compounded monthly. For the remaining two years, the interest rate changes to 13% per annum compounded half-yearly.
   (a) By using the nominal rates, calculate the amount of money Justine received as a birthday gift.
   (b) Convert the nominal rates to effective annual rates.
   (c) By using the effectives rates, calculate the amount of money Justine received as a birthday gift. What do you notice?

Lesson 44

FINANCIAL MATHEMATICS (4)

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.
Assessment Standard
Demonstrate an understanding of different periods of compounding growth and decay.

Overview

In this lesson you will learn about:

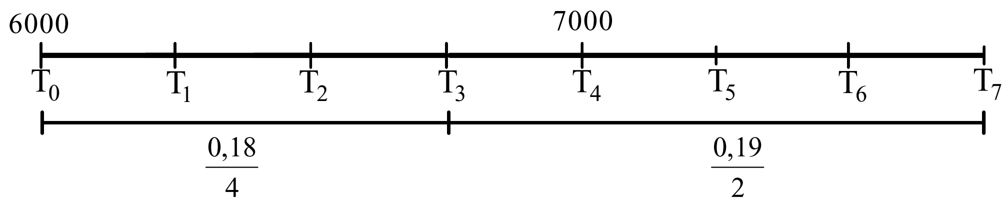
- Future and Present Value calculations involving Additional Payments and Withdrawals.

DVD

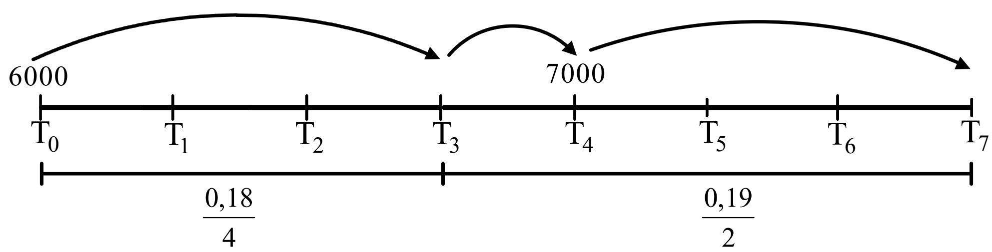
Lesson

Example 1

R6 000 is deposited into a savings account. Four years later, R7 000 is added to the savings. The interest rate for the first three years is 18% per annum compounded quarterly. Thereafter, the interest rate changes to 19% per annum compounded semi-annually. Calculate the value of the savings at the end of the seventh year.



Method 1 (Moving from L $\rightarrow$ R on the time line)



Value at T_3:

$$6000\left(1 + \frac{0,18}{4}\right)^{12}$$
$$= 10\ 175,2886$$

Value at T_4:

$$10\ 175,2886\left(1+ \frac{0,19}{2}\right)^{2} + 7000$$
$$= 19\ 200,42541$$

Value at T_7:

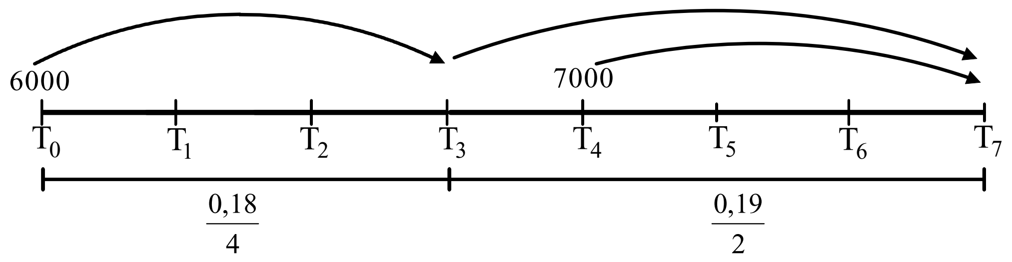
$$19\ 200,42541\left(1+ \frac{0,19}{2}\right)^{6}$$
$$= R33\ 097,53$$

Method 2 (No partial calculation)

$$A = \left\{6000\left(1 + \frac{0,18}{4}\right)^{12}\left(1 + \frac{0,19}{2}\right)^{8} + 7000\right\}\left(1 + \frac{0,19}{2}\right)^{6}$$

A = R33 097,53

Method 3 (Recommended) (Moving each entry to T_7)



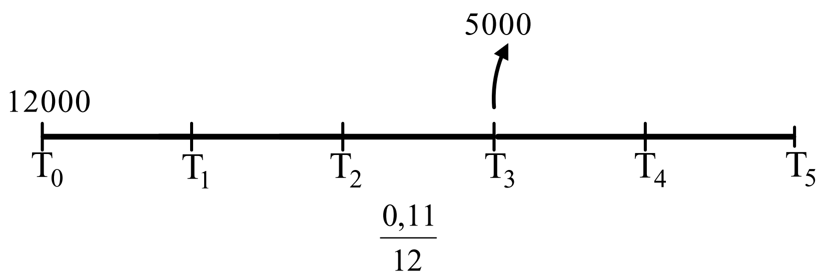
$$A = 6000\left(1 + \frac{0,18}{4}\right)^{12}\left(1 + \frac{0,19}{2}\right)^{8} + 7000\left(1 + \frac{0,19}{2}\right)^{6}$$

A = R33 097,53

Link to Activity 1

Example 2

Peter invests R12 000 in a savings account in order to save up for an overseas trip in five years time. The interest rate for the five-year period is 11% per annum compounded monthly. At the end of the third year, he runs into financial difficulty and withdraws R5 000 from the account. How much money will he have saved at the end of the five-year period?



Method (Recommended)

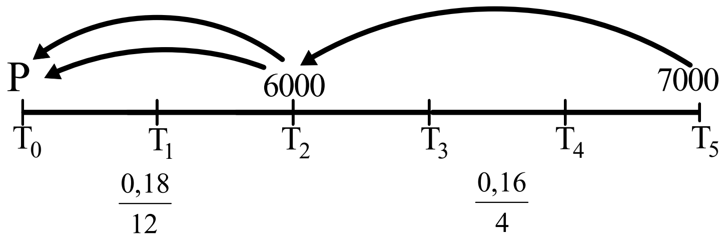
$$A = 12000\left(1 + \frac{0,11}{12}\right)^{60} - 5000\left(1 + \frac{0,11}{12}\right)^{24}$$

$\therefore$ A = R14 522,85

Link to Activity 2

Example 3

Sean takes out a loan. He repays the loan by means of a payment of R6 000 two years after the granting of the loan. Three years later, he repays a final amount of R7 000. The interest rate during the first two years of the loan is 18% per annum compounded monthly. For the remaining three years, the interest rate changes to 16% per annum compounded quarterly. How much money did Sean originally borrow?



$P = 6000\left(1+\frac{0,18}{12}\right)^{24} + 7000\left(1+\frac{0,16}{4}\right)^{-12}\cdot\left(1+\frac{0,18}{12}\right)^{-24}$

$\therefore$ P = 4197,263517 + 3058,531477

$\therefore$ P = R7255,79

Note: 4197,263517 is the present value of 6 000 at T_0 (interest is removed from 6 000 to get 4197,263517).

3058,531477 is the present value of 7 000 at T_0 (interest is removed from 7 000 to get 3058,531477).

If you now add the interest free amounts, you get the value of the original loan.

Link to Activity 3

Activity 1

1. R3 500 is deposited into a savings account. Three years later, R4 000 is added to the savings. The interest rate for the two years is 8% per annum compounded monthly. Thereafter, the interest rate changes to 10% per annum compounded semi-annually. Calculate the value of the savings at the end of the sixth year.

2. Thabo deposits R5 000 into a savings account. Three years later, he deposits a further R4 000 into the account. Four years after this, he deposits a further R6 000 into the account. The interest rate for the first 4 years is 13% per annum compounded semi-annually. For the next 3 years, the interest rate increases to 14% per annum compounded quarterly. Calculate the future value of the savings at the end of the seven year period.

3. It is the 1st January 2007. Leah decides to deposit R2 000 into a savings account at the end of the year. At the end of 2008, she deposits double her first amount into the account. At the end of the 2009, she deposits double her second amount into the account. The interest rate for the first two years (2007-2008) is 9% per annum compounded monthly. The interest rate for the remaining two years (2009-2010) is 8% per annum compounded monthly. Calculate the value of her investment at the end of 2010.

4. Joseph deposits Rx into a savings account. Two years later, he deposits a further R$2x$ into the account. Three years after this, he deposits R$3x$ into the account. The interest rate for the five years is 18% per annum compounded monthly. He receives R60 000 at the end of the five year period. Calculate the value of x.

Activity 2

1. A man invests R15 000 in a savings account in order to save up for an overseas trip in six years time. The interest rate for the six-year period is 13% per annum compounded half-yearly. At the end of the fourth year, he runs into financial difficulty and withdraws R3 000 from the account. How much money will he have saved at the end of the six-year period?
2. Moira starts a savings account and deposits R5 000 into the account. Twenty four months later, she deposits a further R6 000 into the account. Three years later, she withdraws R3 000 to buy a gift for her husband. The interest rate during the first three years is 8% per annum compounded monthly. The interest rate then changes to 9% per annum compounded quarterly. Calculate the value of Moira's investment two years after her withdrawal of R3 000.
3. Michelle deposits R4 300 into a savings account. Three years later, she deposits a further R7 000 into the account. Two years later, she withdraws R2 000. Two years after this, she deposits a final amount of R1 000 into the account. The interest rate during the first three years is 13% per annum compounded monthly. For the remaining four years, the interest rate changes to 14% per annum compounded quarterly. Calculate the future value of the savings at the end of the seven year period.

Activity 3

1. A certain amount deposited in a savings account now will grow to R13 000 after 7 years. The interest rate during the first 4 years is 9% per annum compounded monthly and for the remaining 3 years is 12% per annum compounded quarterly. How much is this amount?
2. A seven-year loan was repaid by means of R2 300 four years after the granting of the loan and R4 200 three years thereafter. The interest rate for the first five years was 10% per annum compounded half-yearly and 12% per annum compounded monthly for the remaining period. What was the amount borrowed?
3. Samantha buys a car and pays an initial deposit of R6 000, R8 000 after 2 years and a further R9 000 two years later. Interest is 18% per annum compounded monthly for the first two years. For the next two years, the interest rate is 19% per annum compounded monthly. Calculate the original price of the car.
4. Thabo pays a deposit of R30 000 for a new car. He repays the remaining loan by paying a further R20 000 in 2 years' time and R100 000 6 years thereafter. Interest is 18% p.a. compounded monthly during the first 3 years, and, 32% p.a. effective for the remaining 5 years. What did Mbeki originally pay for the car?

Lesson 45

TRANSFORMATION GEOMETRY (1)

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape and Space
Assessment Standard
Investigate, generalise and apply the effect on the coordinates of:
- The point after rotation around the origin through an angle of 90°.

Overview

In this lesson you will:

- Revise translations and reflections from Grade 10.
- Learn about rotations of 90° clockwise and anti-clockwise.

Lesson

REVISION OF TRANSLATIONS AND REFLECTIONS (GRADE 10)

TRANSLATIONS

A **translation** is a horizontal or vertical "slide" from one position to another. The object translated doesn't change its shape or orientation.

If the point $(x; y)$ is translated to the point $(x + a; y + b)$ where a is a horizontal translation and b is a vertical translation then:
if $a > 0$, the horizontal translation is to the right.
if $a < 0$, the horizontal translation is to the left.
if $b > 0$, the vertical translation is upward.
if $b < 0$, the vertical translation is downward.

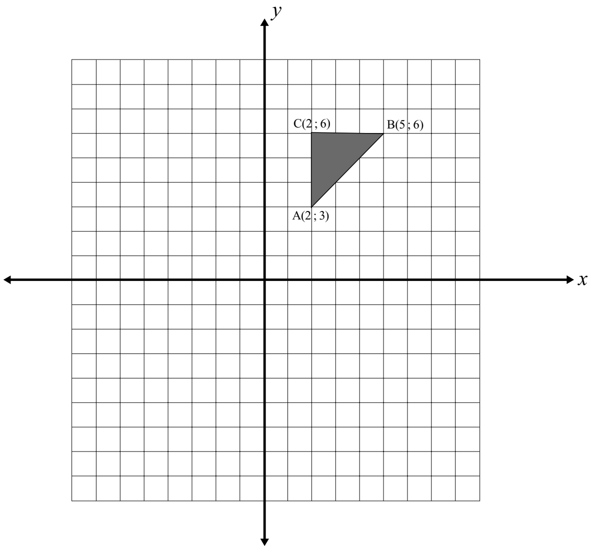
REFLECTIONS

A **reflection** is a mirror image of a shape about a line of reflection.

Summary of the rules for reflection
Reflection about the y-axis:
$(x; y) \rightarrow (-x; y)$ (The first coordinates differ in sign)
Reflection about the x-axis:
$(x; y) \rightarrow (x; -y)$ (The second coordinates differ in sign)
Reflection about the line $y = x$:
$(x; y) \rightarrow (y; x)$ (The first and second coordinates have interchanged)

Example 1

Consider $\triangle$ABC in the figure below with the given coordinates.



(a) Draw the image of $\triangle$ABC under the transformation rule $(x; y) \rightarrow (x - 8; y + 2)$. Call the image $\triangle A'B'C'$.

Describe the transformation in words.

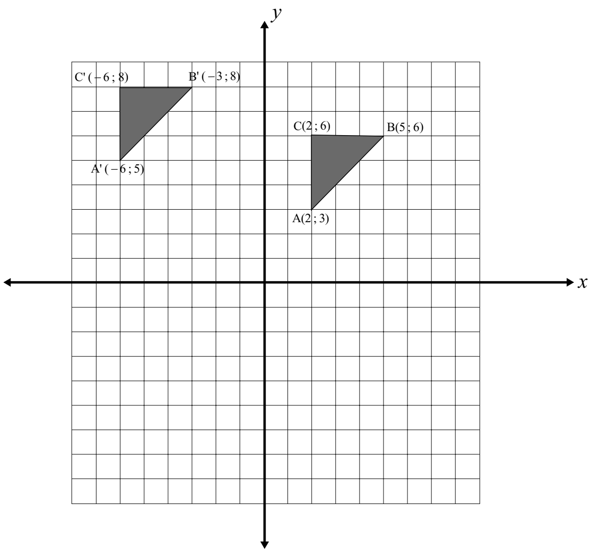
This transformation is a translation of 8 units to the left (horizontally) and then 2 units upwards (vertically).

The coordinates of the image $\triangle A'B'C'$ can be calculated by substituting the coordinates of A, B and C into the rule $(x; y) \rightarrow (x - 8; y + 2)$ as follows:

$A(2; 3) \rightarrow A'(2 - 8; 3 + 2) = A'(-6; 5)$

$B(5; 6) \rightarrow B'(5 - 8; 6 + 2) = B'(-3; 8)$

$C(2; 6) \rightarrow C'(2 - 8; 6 + 2) = C'(-6; 8)$



(b) Draw the image of $\triangle A'B'C'$ under the transformation rule $(x; y) \rightarrow (x; -y)$. Call the image $\triangle A''B''C''$.

Describe the transformation in words.

This transformation is a reflection about the x-axis.

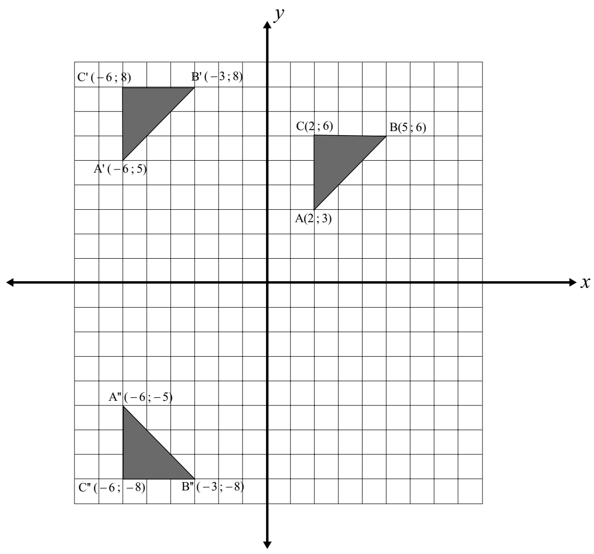
The y-coordinates of the image differ in sign from the original figure.

The coordinates of the image $\triangle A''B''C''$ can be calculated by substituting the coordinates of A', B' and C' into the rule $(x; y) \to (x; -y)$ as follows:

A' (–6; 5)$\to$ A'' (–6; –5)

B' (–3; 8)$\to$ B'' (–3; –8)

C' (–6; 8)$\to$ C'' (–6; –8)



(c) Draw the image of $\triangle A''B''C''$ under the transformation rule $(x; y) \to (-x; y)$. Call the image $\triangle A'''B'''C'''$.

Describe the transformation in words.

This transformation is a reflection about the y-axis.

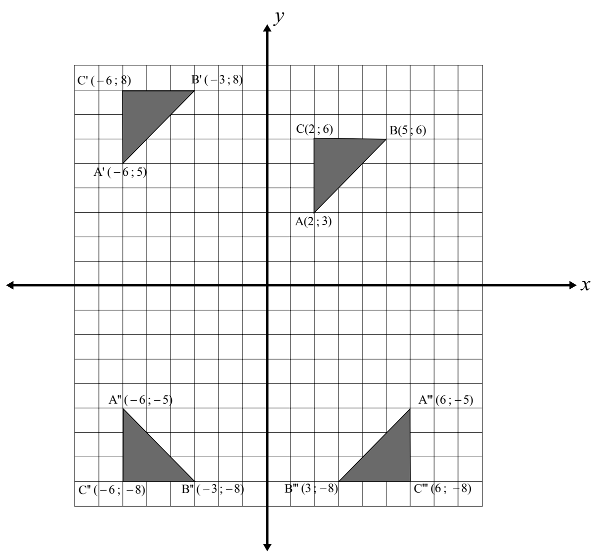
The x-coordinates of the image differ in sign from the original figure.

The coordinates of the image $\triangle A'''B'''C'''$ can be calculated by substituting the coordinates of A', B' and C' into the rule $(x; y) \to (-x; y)$ as follows:

A'' (–6; –5)$\to$ A''' (6; –5)

B'' (–3; –8)$\to$ B''' (3; –8)

C'' (–6; –8)$\to$ C''' (6; –8)

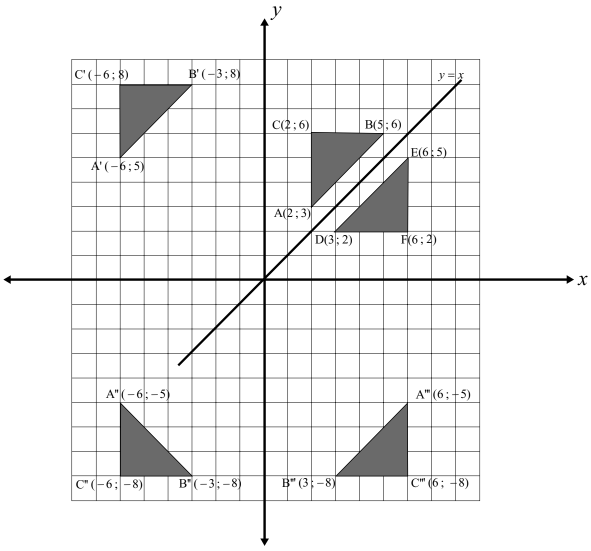


(d) Draw the image of $\triangle ABC$ under the transformation rule $(x; y) \to (y; x)$. Call the image $\triangle DEF$.

Describe the transformation in words.

This transformation is a reflection about the line $(y = x)$.

The x and y coordinates of the original are interchanged in the image points.

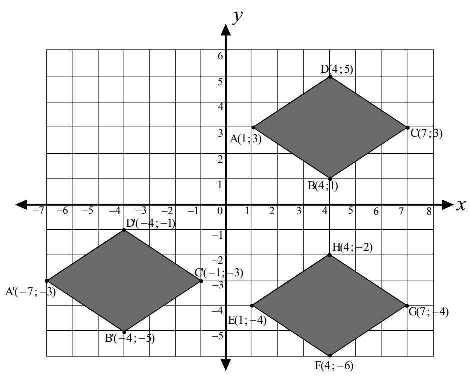


Activity 1

PAIRS

BASELINE ASSESSMENT

QUESTION 1



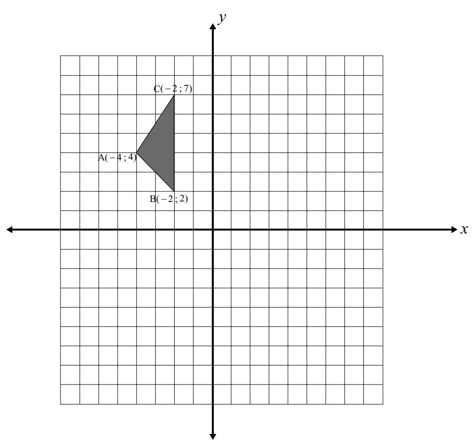
Region EFGH can been seen as the image of either region ABCD or region A′B′C′D′.

Determine the transformation rule if:

(a) EFGH is the image of ABCD

(b) EFGH is the image of A′B′C′D′

QUESTION 2



On the set of axes provided alongside, draw the image of $\triangle ABC$ under each of the following transformations:

(a) $(x; y) \to (-x; y)$

(b) $(x; y) \to (x; -y)$

(c) $(x; y) \to (y; x)$

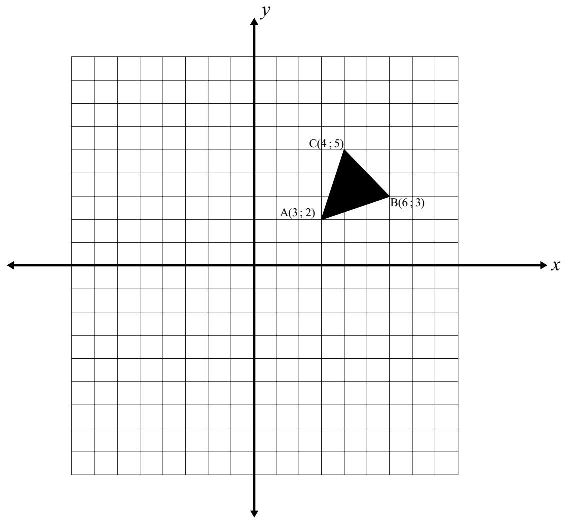
(d) $(x; y) \to (x + 8; y)$

Lesson

ROTATIONS OF 90° CLOCKWISE AND ANTI-CLOCKWISE

Rotations of 90° anti-clockwise

Consider △ABC with vertices A(3; 2), B(6; 4) and C(4; 5).

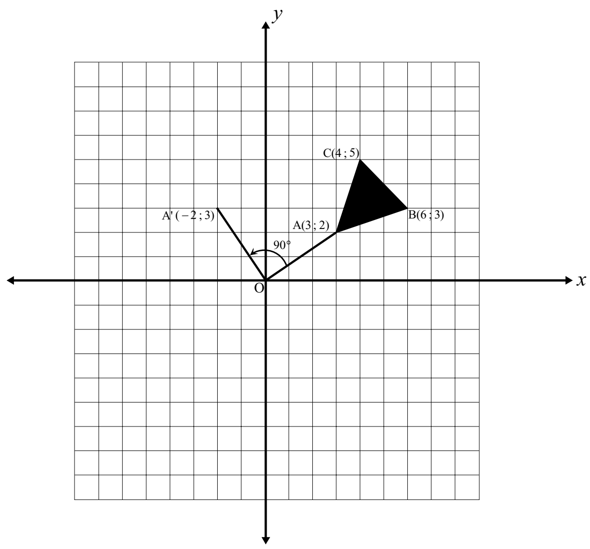


We will now rotate △ABC 90° in an anti-clockwise direction by using the following approach:

(a) Join the origin to point A by means of a line.

(b) Measure the length of line OA.

(c) Use a protractor and a ruler to construct line OA′, which is the rotation of the line OA by 90° anti-clockwise.



Then do the same with line OB:

(d) Join the origin to point B by means of a line.

(e) Measure the length of line OB.

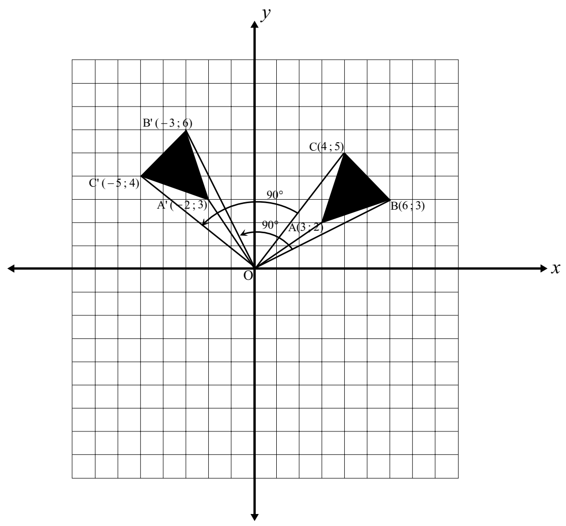
(f) Use a protractor and a ruler to construct line OB′, which is the rotation of the line OB by 90° anti-clockwise.

Then do the same with line OC:

(g) Join the origin to point C by means of a line.

(h) Measure the length of line OC.

(i) Use a protractor and a ruler to construct line OC′, which is the rotation of the line OC by 90° anti-clockwise.



Now complete the following table:

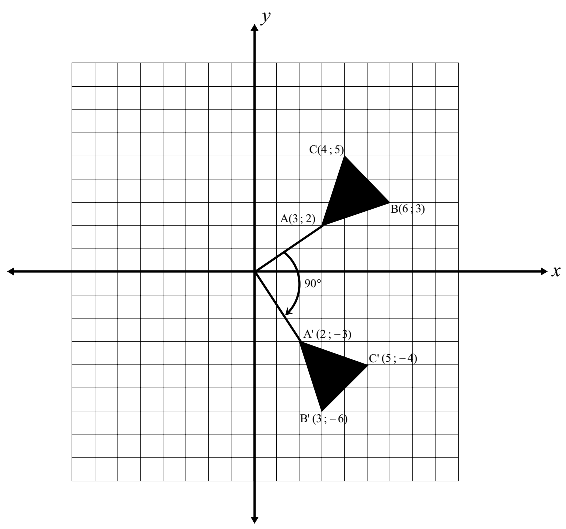
Point	*x*-coordinate	*y*-coordinate
A	3	2
A′	–2	3
B	6	3
B′	–3	6
C	4	5
C′	–5	4

The algebraic rule for rotating a point $(x; y)$ 90° in an anti-clockwise direction is given by: $(x; y) \rightarrow (-y; x)$

Rotations of 90° clockwise

Using the same approach, rotate △ABC 90° in a clockwise direction.

Draw the newly formed triangle and then complete the table which follows.



Now complete the following table:

Point	x-coordinate	y-coordinate
A	3	2
A′	2	–3
B	6	3
B′	3	–6
C	4	5
C′	5	–4

The algebraic rule for rotating a point $(x; y)$ 90° in a clockwise direction

is given by: $(x; y) \rightarrow (y; -x)$

SUMMARY OF THE 90° ROTATION RULES

Rotation of 90° anti-clockwise:

$(x; y) \rightarrow (-y; x)$

To get the coordinates of the image points:

- First swop around the first and second coordinates.
- Then change the sign of the newly formed first coordinate.

Rotation of 90° clockwise:

$(x; y) \rightarrow (y; -x)$

To get the coordinates of the image points:

- First swop around the first and second coordinates.
- Then change the sign of the newly formed second coordinate.

Example 1

(a) Determine the coordinates of the image of the point A(2; 5) under the following transformations. Describe each transformation.

(1) $(x; y) \rightarrow (-y; x)$

(2) $(x; y) \rightarrow (y; -x)$

Solution

(1) A (2; 5) A′(–5; 2) Rotation of 90° anti-clockwise

(2) A (2; 5) A′(5; –2) Rotation of 90° clockwise

(b) Determine the coordinates of the image of the point A(–2; 5) under the following transformations. Describe each transformation.

(1) $(x; y) \rightarrow (-y; x)$

(2) $(x; y) \rightarrow (y; -x)$

Solution

(1) A (–2; 5) → A′(–5; –2) Rotation of 90° anti-clockwise

(2) A (–2; 5) → A′(5; 2) Rotation of 90° clockwise

(c) Determine the coordinates of the image of the point A(– 2; – 5) under the following transformations. Describe each transformation.

(1) $(x; y) \rightarrow (-y; x)$

(2) $(x; y) \rightarrow (y; -x)$

Solution

(1) $A(-2; -5) \rightarrow A'(5; -2)$ Rotation of 90° anti-clockwise

(2) $A(-2; -5) \rightarrow A'(-5; 2)$ Rotation of 90° clockwise

(d) Determine the coordinates of the image of the point A(2; –5) under the following transformations. Describe each transformation.

(1) $(x; y) \rightarrow (-y; x)$

(2) $(x; y) \rightarrow (y; -x)$

Solution

(1) $A(2; -5) \rightarrow A'(5; 2)$ Rotation of 90° anti-clockwise

(2) $A(2; -5) \rightarrow A'(-5; -2)$ Rotation of 90° clockwise

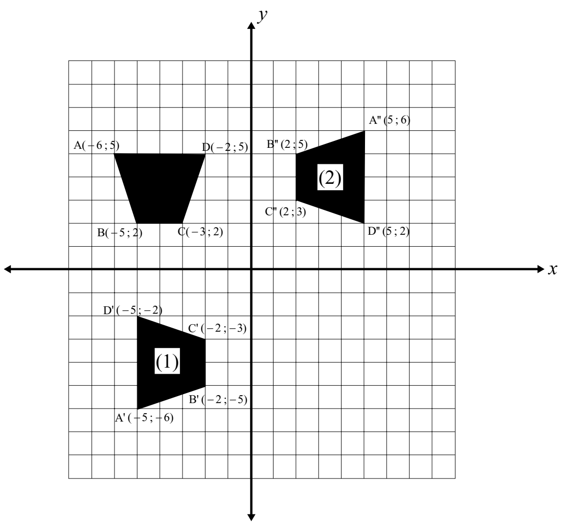
Example 2

Draw the image of the shaded figure ABCD under the following transformations. Describe each transformation.

(1) $(x; y) \rightarrow (-y; x)$

(2) $(x; y) \rightarrow (y; -x)$

Solution



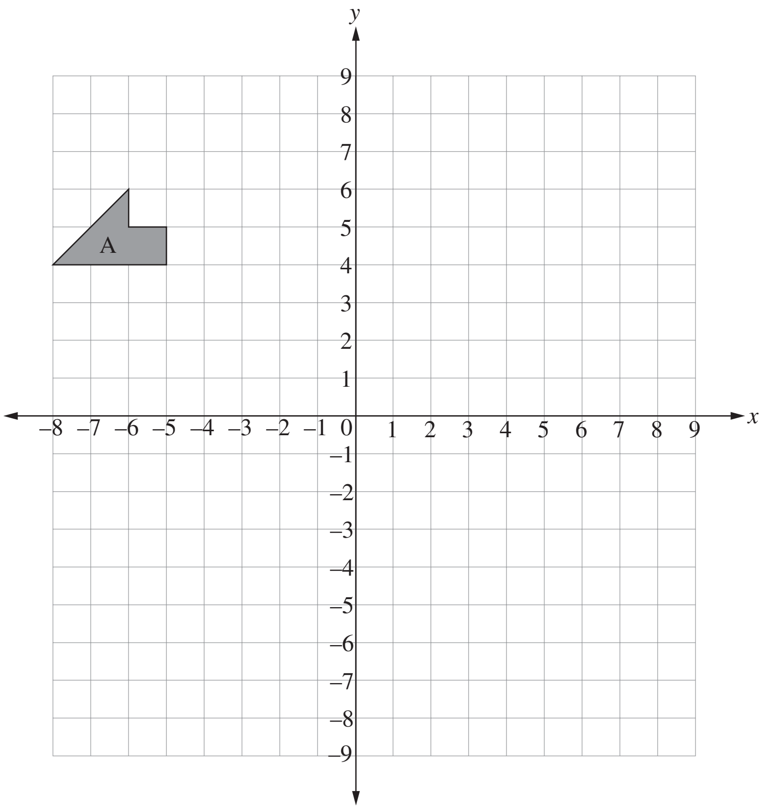
INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 2

On the axes provided below, draw the images of the shaded figure A, under the following transformations. Describe each of the transformations in words.

1. $(x; y) \rightarrow (-x; y)$ (Call the image B)
2. $(x; y) \rightarrow (x; -y)$ (Call the image C)
3. $(x; y) \rightarrow (y; x)$ (Call the image D)
4. $(x; y) \rightarrow (y; -x)$ (Call the image E)
5. $(x; y) \rightarrow (-y; x)$ (Call the image F)



TRANSFORMATION GEOMETRY (2)

Lesson 46

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape and Space
Assessment Standard
Investigate, generalise and apply the effect on the coordinates of:

- The point after rotation around the origin through an angle of 90° or 180°.
- The vertices of a polygon after enlargement through the origin by a constant factor.

Overview

In this lesson you will:

- Learn about rotations of 180° clockwise and anti-clockwise.
- Learn about enlargements and reductions.

Lesson

SUMMARY OF THE 90° ROTATION RULES

Rotation of 90° anti-clockwise:

$(x; y) \rightarrow (-y; x)$

To get the coordinates of the image points:

- First swop around the first and second coordinates.
- Then change the sign of the newly formed first coordinate.

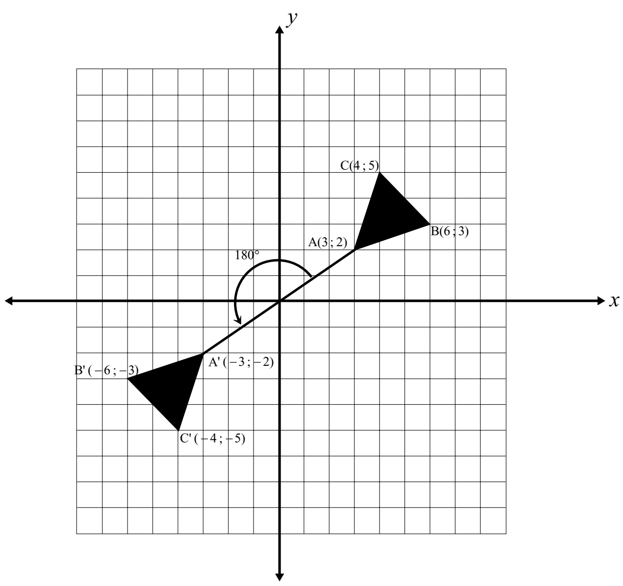
Rotation of 90° clockwise:

$(x; y) \rightarrow (-y; -x)$

To get the coordinates of the image points:

- First swop around the first and second coordinates.
- Then change the sign of the newly formed second coordinate.

ROTATIONS OF 180° CLOCKWISE AND ANTI-CLOCKWISE



Using the same approach as in the previous episode, rotate △ABC 180° in a clockwise (or anti-clockwise) direction. Draw the newly formed triangle and then complete the table which follows.

Now complete the following table:

Point	***x*-coordinate**	***y*-coordinate**
A		
A′		
B		
B′		
C		
C′		

The algebraic rule for rotating the point $(x; y)$ 180° in a clockwise or anti-clockwise direction is given by $(x; y) \rightarrow (-x; -y)$.

SUMMARY OF THE 180° ROTATION RULE

Rotation of 180° clockwise or anti-clockwise: $(x; y) \rightarrow (-x; -y)$

To get the coordinates of the image points:

- Change the signs of the original first and second coordinates.

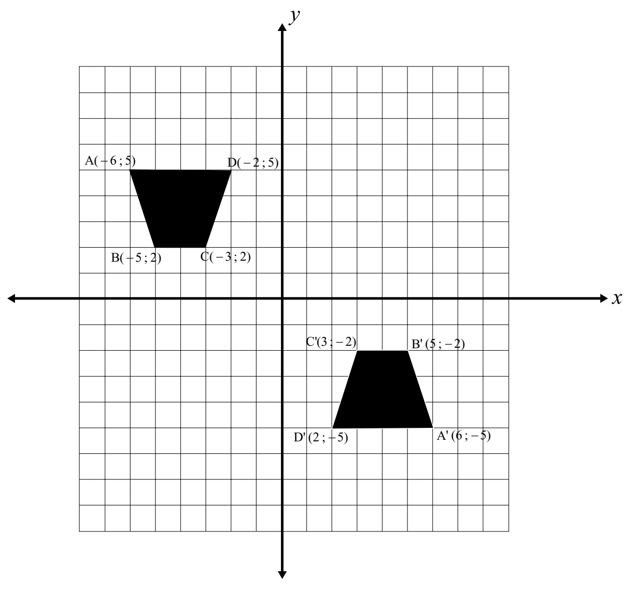
Example 1

Determine the coordinates of the image of the following points under the transformation $(x; y) \rightarrow (-x; -y)$.

(1) A(2; 5) (2) B(–3; 4)

(1) A(2; 5) → A′ (–2; –5) (2) B(–3; 4) → B′(3; –4)

Example 2



Draw the image of figure ABCD under the rule $(x; y) \rightarrow (-x; -y)$.

Describe the transformation.

Rotation of 180°

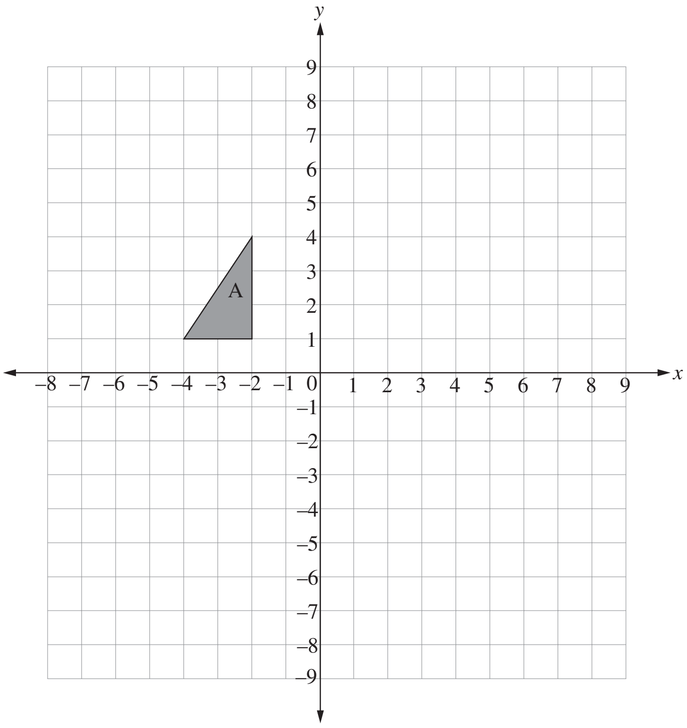
Activity 1

PAIRS

FORMATIVE ASSESSMENT

On the axes provided below, draw the images of the shaded figure A, under the following transformations. Describe each transformation.

1. $(x; y) \rightarrow (y; -x)$ (Call the image B)
2. $(x; y) \rightarrow (-y; x)$ (Call the image C)
3. $(x; y) \rightarrow (-x; -y)$ (Call the image D)

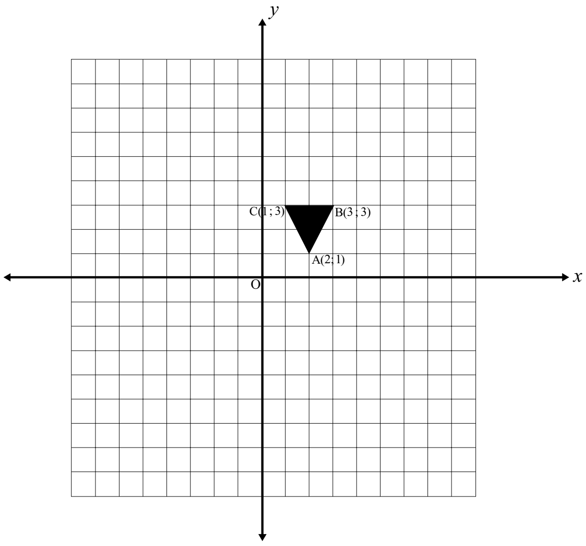


Lesson

ENLARGEMENTS AND REDUCTIONS

A. ENLARGEMENTS OF THE FORM $(x; y) \rightarrow (kx; ky)$ where k>1

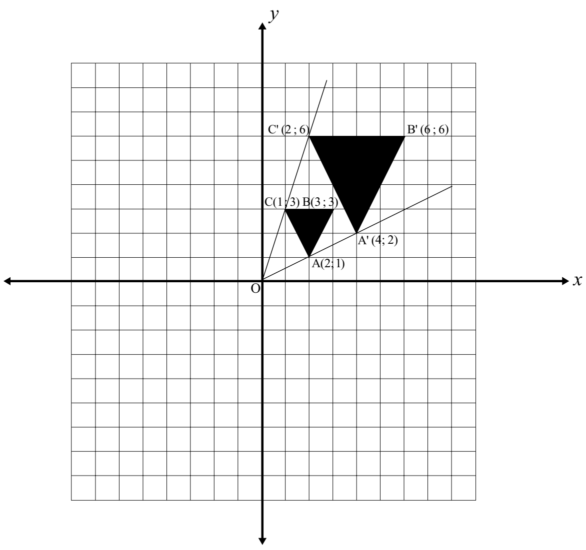
Example 1



Consider $\triangle$ABC with vertices A(2; 1), B(3; 3) and C(1; 3)

We will now enlarge $\triangle$ABC by applying a scale factor of 2 to the coordinates of the vertices using the rule $(x; y) \rightarrow (2x; 2y)$.

The coordinates of the newly formed enlargement of $\triangle$ABC are:



A(2; 1) $\rightarrow$ A′ (4; 2)

B(2; 1) $\rightarrow$ B′ (6; 6)

C(1; 3) $\rightarrow$ C′ (2; 6)

Some interesting facts emerge from this enlargement:

(a) $\triangle ABC ||| \triangle A'B'C'$ since their corresponding sides are in proportion.

This can be verified by calculating the lengths of the sides and showing that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$

(b) $\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle ABC} = \frac{1}{2}(\text{base } B'C')(\text{height})/\frac{1}{2}(\text{base})(\text{height}) = \frac{\frac{1}{2}(4)(4)}{\frac{1}{2}(2)(2)} = 4 = 2^2$

This means that the area of $\triangle A'B'C'$ is 2^2 times larger than the area of $\triangle$ABC.

In general, then, the area of the newly formed triangle under the rule $(x; y) \rightarrow (kx; ky)$ where $k > 1$ is k^2 times larger than the area of the original triangle.

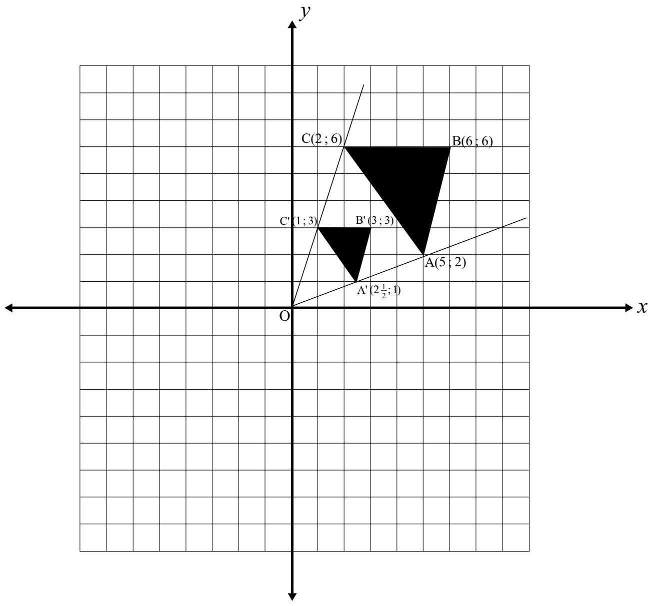
B. REDUCTIONS OF THE FORM $(x; y) \rightarrow (kx; ky)$ where $0 < k < 1$

Example 2

Consider $\triangle$ABC with vertices A(5; 2), B(6; 6) and C(2; 6)

We will now reduce $\triangle$ABC by using the rule $(x; y) \rightarrow \left(\frac{1}{2}x; \frac{1}{2}y\right)$.

The coordinates of the newly formed $\triangle A'B'C'$ are:



$A(5; 2) \rightarrow A'(2\frac{1}{2}; 1)$

$B(6; 6) \rightarrow B'(3; 3)$

$C(2; 6) \rightarrow C'(1; 3)$

C. TRANSFORMATIONS OF THE FORM $(x; y) \rightarrow (kx; ky)$ where $k<0$

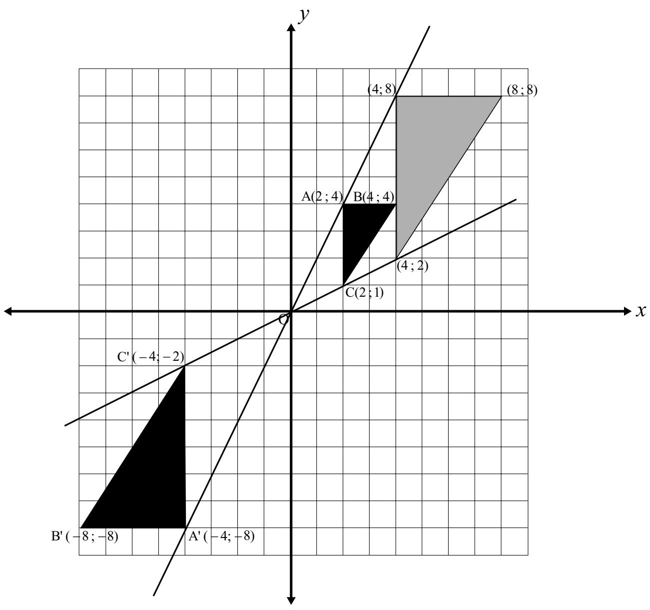
These transformations combine an enlargement or reduction with a rotation of 180°.

Example 3

Draw the image of A(2; 4)B(4; 4) and C(2; 1) under the transformation rule $(x; y) \rightarrow (-2x; -2y)$.

This transformation involves two different types:

- Enlargement by a scale factor of 2: $(x; y) \rightarrow (2x; 2y)$
  Followed by a:
- Rotation of 180°: $(2x; 2y) \rightarrow (-2x; -2y)$



SUMMARY OF THE ENLARGEMENT AND REDUCTION RULES

$(x; y) \rightarrow (\mathrm{k}x; \mathrm{k}y)$

- If k>1, the image is an enlargement of the original figure. Multiply the original first and second coordinates by k units to get the coordinates of the image.
- If 0<k<1, the image is a reduction of the original figure. Multiply the original first and second coordinates by k units to get the coordinates of the image.
- If k<0, the image is a rotation of 180° of the original figure followed by an enlargement of the original figure. Multiply the original first and second coordinates by k units to get the coordinates of the image.

INDIVIDUAL

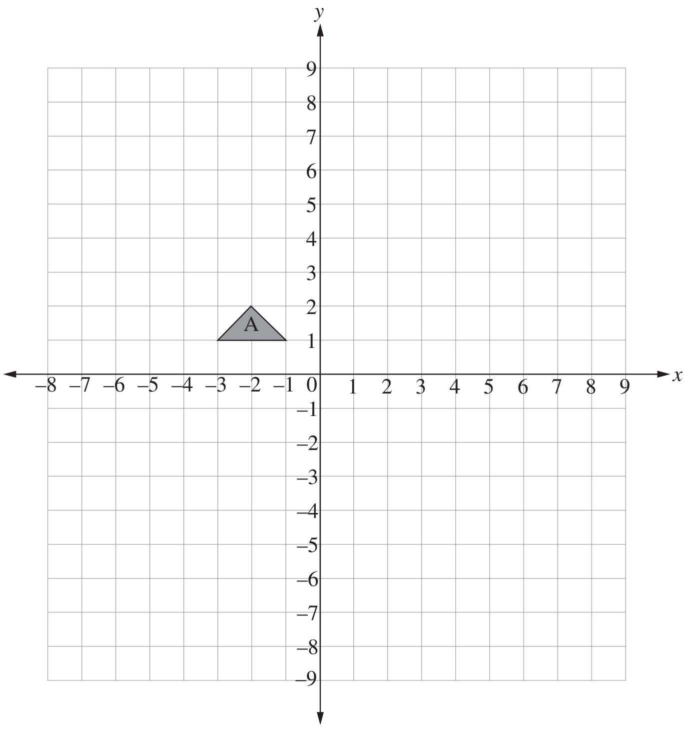
SUMMATIVE ASSESSMENT

Activity 2

On the axes provided below, draw the images of the shaded figure A, under the following transformations.

1. $(x; y) \rightarrow (2x; 2y)$ (Call the image B)
2. $(x; y) \rightarrow (-x; -y)$ (Call the image C)
3. $(x; y) \rightarrow (-2x; -2y)$ (Call the image D)

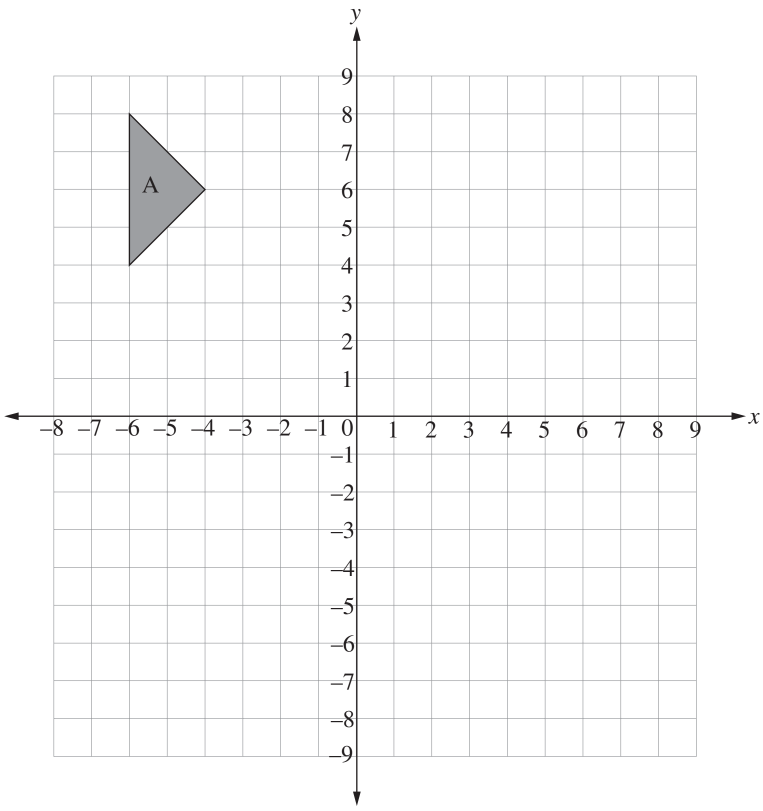
Describe each of the above transformations in words.



Activity 3

On the axes provided below, draw the images of the shaded figure A, under the following transformations.

1. $(x; y) \rightarrow \left(\frac{1}{2}x; \frac{1}{2}y\right)$ (Call the image B)
2. $(x; y) \rightarrow (-x; -y)$ (Call the image C)
4. $(x; y) \rightarrow \left(-\frac{1}{2}x; \frac{1}{2}y\right)$ (Call the image D)
5. $(x; y) \rightarrow (-y; x)$ (Call the image F)
6. $(x; y) \rightarrow (y; -x)$ (Call the image G)



Lesson 47

DATA HANDLING (1)

Learning Outcomes and Assessment Standards

Learning Outcome 4: Data Handling and Probability
Assessment Standard
Calculate and represent measures of central tendency and dispersion in univariate numerical data by:

- Five number summary
- Box and whisker diagrams
- Ogives
- Variance and standard deviation

Overview

In this lesson you will:

- Revise the terms mean, median, mode and quartiles.
- Learn about Five Number Summaries.
- Learn about Box and Whisker Plots.

Lesson

Revision of Grade 10 concepts

THE MEAN

The mean of a set of data is defined as $\overline{x}$ (x bar).

$$\overline{\text{x}} = \frac{\text{sum of the values}}{\text{number of the values}} = \frac{\Sigma x}{n}$$

Example

Calculate the mean of the following data:

12, 13, 13, 15, 16, 16, 16, 16, 16, 17, 17, 18, 18, 18, 18

$$\overline{\text{x}} = \frac{\Sigma x}{\text{n}} = 12+13+13+15+16+16+16+16+16+17+17+18+18+18+18 = \frac{239}{15} = 15{,}9$$

THE MODE

The mode is the most commonly occurring observation.

Example

Class A	1	1	1	2	4	5	7	9	10							
Class B	1	1	1	2	4	5	5	5	5	7	7	8	8	8	9	10

For Class A, the mode is 1. For Class B, the mode is 5.

Consider the following set of marks for a Class C:

Class C	0	1	1	2	2	2	3	4	4	4	5	5	6	7	9	10

There are two modes in this set of data: 2 and 4 (they appear the same number of times and are the most frequently occurring marks. The data is said to be **bimodal**.

QUARTILES

Quartiles are measures of dispersion (or spread) around the median, which is a better measure of central tendency. The median divides the data into two halves. The quartiles further subdivide the data into quarters.

There are therefore three quartiles:

The Lower Quartile (Q_1): This is the median of the lower half of the values. We also call this the 25th percentile.

The Median (Q_2): The value that divides the data into halves. We also call this the 50th percentile.

The Upper Quartile(Q_3): This is the median of the upper half of the values. We also call this the 75th percentile.

Useful formulae to determine the position of the quartiles are:

The Lower Quartile (Q_1): $\frac{1}{4}(n + 1)$

The Median (Q_2):

$\frac{1}{2}(n + 1)$

The Upper Quartile(Q_3):$\frac{3}{4}(n + 1)$

Example 1 (Odd number of values)

Note:
If the number of values (n) in the data set is odd, the median will always be part of the data set.

To find the median we use $\left(\frac{n+1}{2}\right)$

The lower and upper quartiles will be part of the data set if $\frac{1}{4}(n + 1)$ and $\frac{3}{4}(n + 1)$ work out to be whole numbers.

The lower and upper quartiles will not be part of the data set if $\frac{1}{4}(n + 1)$ and $\frac{3}{4}(n + 1)$ do not work out to be whole numbers.

(a) Consider the following set of marks obtained on a class test out of 10 marks. The number of marks is odd.

2	2	**3**	4	5	**5**	6	7	**7**	8	9
		Lower Quartile Q_1			Median Q_2			Upper Quartile Q_3		

The position of $Q_2 = \frac{1}{2}(11 + 1) = 6$.

The **Median** of the data is 5 (the 6th value).

The position of $Q_1 = \frac{1}{4}(11 + 1) = 3$

The **Lower Quartile** of the data is 3 (the 3rd value). It is a part of the data set.

The position of $Q_3 = \frac{3}{4}(11 + 1) = 9$

The **Upper Quartile** of the data is 7 (the 9th value). It is part of the data set.

(b) Consider the following set of 13 marks obtained on a class test out of 10 marks:

2	3	4	5	5	5	6	7	7	8	9	10	10

The position of $Q_2 = \frac{1}{2}(13 + 1) = 7$th position.

The **Median** of the data is the 7th value:

$Q_2 = 6$

The position of $Q_1 = \frac{1}{4}(13 + 1) = 3,5$th position (In the middle of point 3 and 4)

The **Lower Quartile** of the data is the average between the 3rd and 4th value:

$Q_1 = \frac{4 + 5}{2} = 4,5$ (not part of the data set)

The position of $Q_3 = \frac{3}{4}(13 + 1) = 10,5$th position

The **Upper Quartile** of the data is the average between the 10th and 11th value:

$Q_3 = \frac{8 + 9}{2} = 8,5$ (not part of the data set)

2	3	4	4,5	5	5	5	6	7	7	8	8,5	9	10	10
			Q_1				Q_2				Q_3			

Example 2 (Even number of values)

Note:
If n is even, the median will not be part of the data set.

If n is even and $\frac{n}{2}$ is even, the lower and upper quartiles will not be values in the data set. Round off the position values up or down to the nearest whole number.

If n is even and $\frac{n}{2}$ is odd, the lower and upper quartiles will be values in the data set.

(a) Consider the following set of 12 marks obtained by a class on a class test out of 100 marks. The number of marks is even.

20	32	43	54	55	61	73	78	89	90	91	98

The position of $Q_2 = \frac{1}{2}(12 + 1) = 6,5$ (average of the 6th and 7th value).

The **Median** of the data is $Q_2 = \frac{61 + 73}{2} = 67$

Since n is even and since $\frac{n}{2} = \frac{12}{2} = 6$ which is even, the lower and upper quartiles will not be values in the data set.

The position of $Q_1 = \frac{1}{4}(12 + 1) = 3,25$ (average of the 3rd and 4th value).

The **Lower Quartile** of the data is $Q_1 = \frac{43 + 54}{2} = 48,5$

The position of $Q_3 = \frac{3}{4}(12 + 1) = 9,75$ (average of the 9th and 10th value).

The **Upper Quartile** of the data is $Q_3 = \frac{89 + 90}{2} = 89,5$

20	32	43	48,5	54	55	61	67	73	78	89	89,5	90	91	98
			Lower Quartile 48,5				Median 67				Upper Quartile 89,5			

(b) Consider the following set of 10 marks obtained by a class on a class test out of 150 marks. The number of marks is even.

12	60	95	105	120	125	130	135	140	142

The position of $Q_2 = \frac{1}{2}(10 + 1) = 5,5$th position (average of the 5th and 6th value).

The **Median** of the data is $\frac{120 + 125}{2} = 122,5$

Since n is even and since $\frac{n}{2} = \frac{10}{2} = 5$ which is odd, the lower and upper quartiles will be values in the data set.

The position of $Q_2 = \frac{1}{4}(10 + 1) = 2,75$ (Round up to the 3rd value)

The **Lower Quartile** of the data is 95

The position of $Q_3 = \frac{3}{4}(10 + 1) = 8{,}25$ (Round down to the 8th value).

The **Upper Quartile** of the data is 135

12	60	95	105	120	122,5	125	130	135	140	142

INTERQUARTILE RANGE (IQR)

The difference between the lower and upper quartile is called the Interquartile Range. It is a better measure of dispersion than the range because it is not affected by extreme values. It is based on the middle half of the data. It indicates how densely the data in the middle is spread around the median. Consider the previous example.

12	60	95	105	120	122,5	125	130	135	140	142

The Interquartile Range (IQR)= $Q_3 - Q_1 = 135 - 95 = 40$

SEMI-INTERQUARTILE RANGE

The semi-interquartile range is half of the interquartile range.

The semi-IQR for the previous example is $\frac{Q3 - Q_1}{2} = \frac{135 - 95}{2} = \frac{40}{2} = 20$.

Activity 1

PAIRS

BASELINE ASSESSMENT

1. For each set of data, determine the quartiles:

A	2	3	5	7	9	10	11	13	15	16	16	17	18	19	21	22	23	25	32	
B	2	3	5	7	9	10	11	13	15	16	16	17	18	19	21	22	23			
C	2	3	5	7	9	10	11	13	15	16	16	17	18	19	21	22	23	25	32	34
D	2	3	5	7	9	10	11	13	15	16	16	17	18	19	21	22	23	25		

2. Class results for a test out of 30 are recorded in the table below.

10A	16	12	16	11	14	15	22	16	17	15	26	23	16	22	16	17	24	19	16	–
10B	20	19	14	10	14	9	8	13	14	30	27	23	24	28	17	29	20	16	14	18
10C	5	20	14	12	7	2	12	21	14	26	14	14	12	14	21	24	14	14	–	–

   (a) Calculate the mean for each class.

   (b) Calculate the mode for each class.

   (c) Calculate the median for each class.

   (d) Calculate the range for each class.

   (e) Calculate the lower quartile for each class.

   (f) Calculate the upper quartile for each class.

   (g) Calculate the interquartile range for each class.

   (h) Calculate the semi-interquartile range for each class.

3. A teacher has recorded the test marks of forty grade 10 learners. The test was out of 10. Draw a frequency table and then calculate the mean, median and mode for this data.

1	9	10	4	7	4	4	10
7	3	9	3	8	9	3	7
7	3	9	4	5	8	6	6
1	3	10	2	2	7	8	7
7	2	7	6	2	8	7	6

Lesson

Five Number Summaries and Box and Whisker Plots

Five Number Summaries and Box and Whisker plots help us to represent and analyse the spread of data about the median.

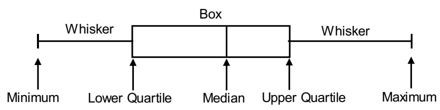
FIVE NUMBER SUMMARIES

The Five Number Summary uses the following measures of dispersion:

- Minimum: The smallest value in the data
- Lower Quartile: The median of the lower half of the values
- Median: The value that divided the data into halves
- Upper Quartile: The median of the upper half of the values
- Maximum: The largest value in the data

BOX AND WHISKER PLOTS

A Box and Whisker Plot is a graphical representation of the Five Number Summary.



Note:

- Half of the values lie between the minimum value and the median.
- Half of the values lie between the median and the maximum value.
- One quarter of the values lies between the minimum value and the lower quartile.
- One quarter of the values lies between the lower quartile and the median.
- One quarter of the values lies between the median and the upper quartile.
- One quarter of the values lies between the upper quartile and the maximum value.
- Half of the values lie between the lower quartile and upper quartile.

Example

Consider the following set of marks for a class test (out of 10) for three classes.

CLASS A	1	1	2	2	3	3	4	4	4	6	7	8	8	9	10	10	10
CLASS B	1	2	4	4	4	4	5	7	8	8	8	8	9	9	9	10	10
CLASS C	1	2	3	3	3	4	5	5	5	6	6	7	7	7	8	9	10

For each class above, create a Five Number Summary and hence a Box and Whisker Plot.

CLASS A	1	1	2	2	2,5	3	3	4	4	4	6	7	8	8	8,5	9	10	10	10

Five Number Summary

Minimum: 1

Lower Quartile (Q_1): Position of $Q_1 = \frac{1}{4}(17 + 1) = 4{,}5$

(average of 4th and 5th value)

$\therefore Q_1 = \frac{2 + 3}{2} = 2{,}5$

Median (M or Q_2): Position of $Q_2 = \frac{1}{2}(17 + 1) = 9$

(9th value)

$\therefore Q_2 = 4$

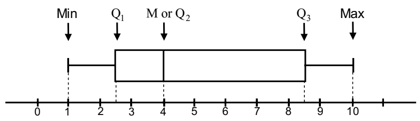
Upper Quartile (Q_3): Position of $Q_3 = \frac{3}{4}(17 + 1) = 13{,}5$

(average of 13th and 14th value)

$\therefore Q_3 = \frac{8 + 9}{2} = 8{,}5$

Maximum: 10

Box and Whisker Plot



CLASS B	1	2	4	4	4	4	4	5	7	8	8	8	8	9	9	9	9	10	10

Five Number Summary

Minimum: 1

Lower Quartile (Q_1): Position of $Q_1 = \frac{1}{4}(17 + 1) = 4{,}5$

(average of 4th and 5th value)

$\therefore Q_1 = \frac{4 + 4}{2} = 4$

Median (M or Q_2): Position of $Q_2 = \frac{1}{2}(17 + 1) = 9$

(9th value)

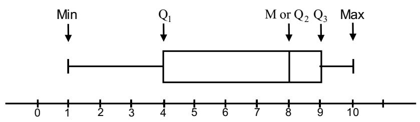
$\therefore Q_2 = 8$

Upper Quartile (Q_3): Position of $Q_3 = \frac{3}{4}(17 + 1) = 13{,}5$

(average of 13th and 14th value)

$\therefore Q_3 = \frac{9 + 9}{2} = 9$

Maximum: 10



Box and Whisker Plot

CLASS C	1	2	3	3	3	3	4	5	5	5	6	6	7	7	7	7	8	9	10

FIVE NUMBER SUMMARY

Minimum: 1

Lower Quartile (Q_1): Position of $Q_1 = \frac{1}{4}(17 + 1) = 4,5$

(average of 4th and 5th value)

$\therefore Q_1 = \frac{3+3}{2} = 3$

Median (M or Q_2): Position of $Q_2 = \frac{1}{2}(17 + 1) = 9$

(9th value)

$\therefore Q_2 = 5$

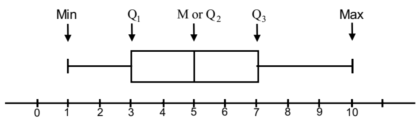
Upper Quartile (Q_3): Position of $Q_3 = \frac{3}{4}(17 + 1) = 13,5$

(average of 13th and 14th value)

$\therefore Q_3 = \frac{7+7}{2} = 7$

Maximum: 10

BOX AND WHISKER PLOT

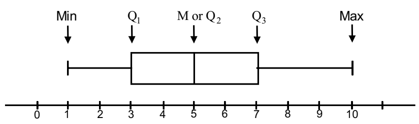


SYMMETRICAL AND SKEWED DATA

- Symmetrical Data Set (relative to the median)

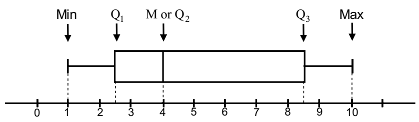
If the data to the left of the median balances with the data on the right, then the data is **symmetrical about the median**.

Consider, for example, CLASS C.

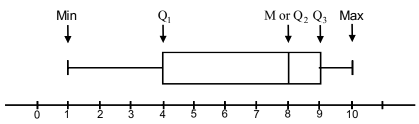


- Skewed Data (relative to the median)

If the data is clustered predominantly to the right of the median, the data is said to be **skewed to the right.** Consider, for example, CLASS A.



If the data is clustered predominantly to the left of the median, the data is said to be **skewed to the left.** Consider, for example, CLASS B.



Activity 2

SUMMATIVE ASSESSMENT

INDIVIDUAL

The number of points scored by four Formula One racing drivers over a number of races are given below:

A	1	1	1	2	6	6	8	8	8	8	10	10	10
B	1	2	6	8	8	8	8	8	8	10	10	10	–
C	1	1	2	2	4	4	6	6	8	8	10	–	–
D	2	2	2	4	4	6	6	8	8	10	10	10	–

(a) Calculate the mean for each of the drivers.

(b) List the Five Number Summary for each driver.

(c) Draw a Box and Whisker plot for each driver.

(d) Discuss each driver's distribution of scores in terms of the spread about the median.

(e) Compare the performance results for each driver by using the information obtained above.

Lesson 48

DATA HANDLING (2)

Learning Outcomes and Assessment Standards

Learning Outcome 4: Data Handling and Probability
Assessment Standard
Calculate and represent measures of central tendency and dispersion in univariate numerical data by:
- Five number summary
- Box and whisker diagrams
- Ogives
- Variance and standard deviation

Overview

In this lesson you will:

- Learn about cumulative frequency graphs (Ogives).

Lesson

Cumulative frequency graphs (ungrouped data)

LC DVD

Determining **cumulative frequencies** is an effective way of representing **ungrouped data**. If you want to find the median of ungrouped data from a frequency table, a useful way to do this is by first determining the cumulative frequencies from the frequency table and then representing the information on what is called a **cumulative frequency graph (or ogive curve)**. The method is illustrated below.

Example 1

Consider the following table which shows the number of learners who obtained certain marks out of 30 for a class test. The marks have been recorded from smallest to largest. The first and second learner in the data each obtained the lowest mark of 18. The last three learners each obtained a mark of 27, which is the highest in the data. The total number of learners is 52.

Marks	18	19	20	21	22	23	24	25	26	27
No of learners	2	2	3	4	6	9	12	6	5	3

We can represent the data in a frequency table and then determine the **cumulative frequencies**.

Marks	**Frequency**	**Cumulative Frequency**	
18	2	2	2 learners got a mark of 8
19	2	(2 + 2) 4	4 learners got a mark of 19 or less
20	3	(4 + 3) 4	7 learners got a mark of 20 or less
21	4	(7 + 4) 11	11 learners got a mark of 21 or less
22	6	(11 + 6) 17	17 learners got a mark of 22 or less
23	9	(17 + 9) 26	26 learners got a mark of 23 or less
24	12	(26 + 12) 38	38 learners got a mark of 24 or less
25	6	(38 + 6) 44	44 learners got a mark of 25 or less
26	5	(44 + 5) 49	49 learners got a mark of 26 or less
27	3	(49 + 3) 52	52 learners got a mark of 27 or less
Total	52		

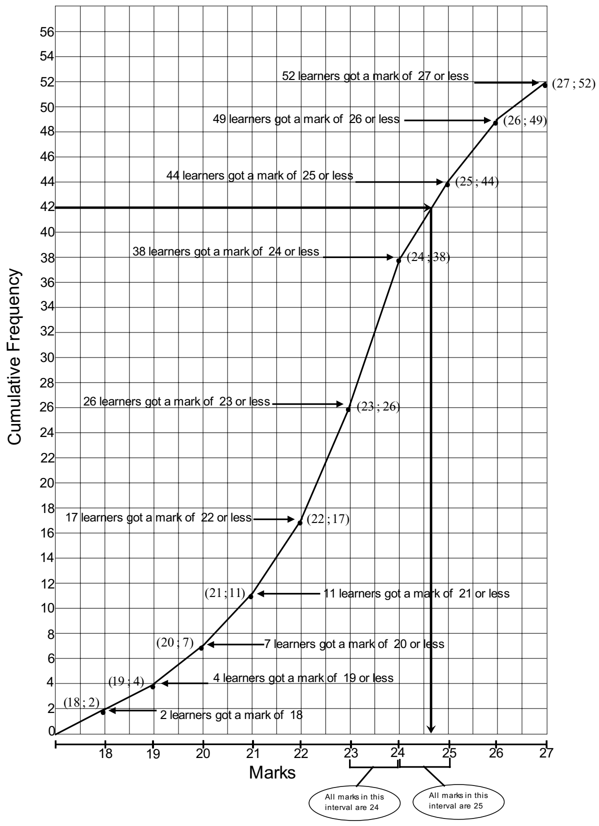
Note:

The total frequency of marks (52) is equal to the final cumulative frequency (52).

We can now represent the information graphically. The graph below is called a **cumulative frequency graph** or **ogive curve**.

The horizontal axis represents the marks. The vertical axis represents the cumulative frequencies of the marks.

The point (19; 4) indicates that 4 learners got a mark of 19 or less.



We can now use the graph to answer the following questions:

(a) Read off from the graph the mark of the 42nd learner.

Draw a horizontal line from the 42nd learner on the vertical axis to the graph. Then draw a vertical line straight down to the horizontal axis. The vertical line meets the interval in which all marks are 25. It is therefore clear that the 42nd learner obtained a mark of 25. This is because the first 44 learners got a mark of 25 or less and the first 38 learners got a mark of 24 or less. The 42nd learner is in the range between the 38th and 44th learner. All learners in this range obtained a mark of 25.

(b) Read off from the graph the mark of the 26th and 27th learner.

The 26th learner got a mark of 23 (see graph)

The 27th learner got a mark of 24 (see graph)

(c) Determine the lower quartile.

Position of $Q_1 = \frac{1}{4}(52 + 1) = 13{,}25$th position

The lower quartile therefore lies between the 13th and 14th learner's mark.

The 13th learner's mark is 22 and the 14th learner's mark is 22 (see graph).

$\therefore\ Q_1 = \frac{1}{2}(22 + 22) = 22$

(d) Determine the median.

Position of $Q_2 = \frac{1}{2}(52 + 1) = 26{,}5$th position

The median therefore lies between the 26th and 27th learner's mark.

The 26th learner's mark is 23 and the 27th learner's mark is 24 (see graph).

$\therefore\ Q_2 = \frac{1}{2}(23 + 24) = 23{,}5$ (the median lies outside the data set)

(e) Determine the upper quartile.

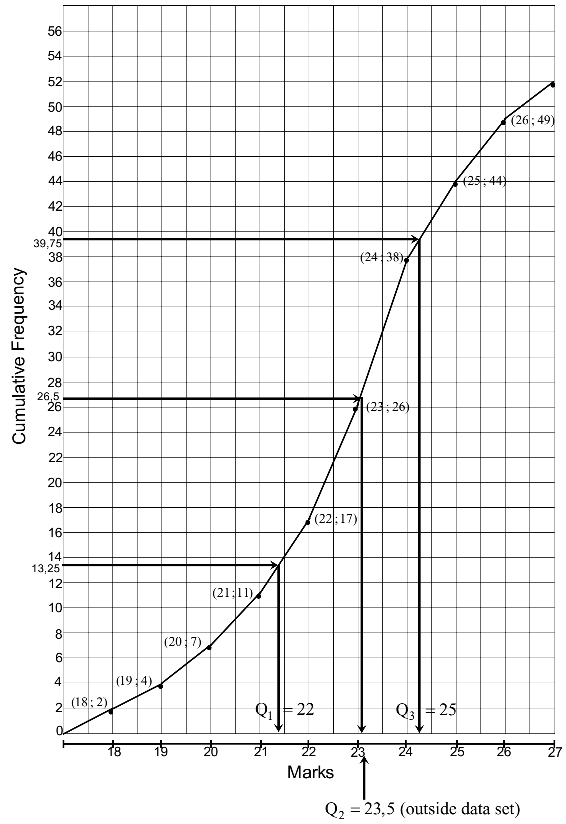
Position of $Q_3 = \frac{3}{4}(52 + 1) = 39{,}75$th position

The upper quartile therefore lies between the 39th and 40th learner's mark.

The 39th learner's mark is 25 and the 40th learner's mark is 25 (see graph).

$\therefore\ Q_3 = \frac{1}{2}(25 + 25) = 25$

The quartiles can be represented on the cumulative frequency graph as follows:



INDIVIDUAL

SUMMATIVE ASSESSMENT

Activity 1

Complete this exercise in this workbook.

The following table contains the number of learners who obtained certain marks on a class test out of 30.

Marks	20	21	22	23	24	25	26	27	28	29
No of learners	3	3	4	5	7	10	13	5	4	2

(a) Draw a cumulative frequency table for this data.

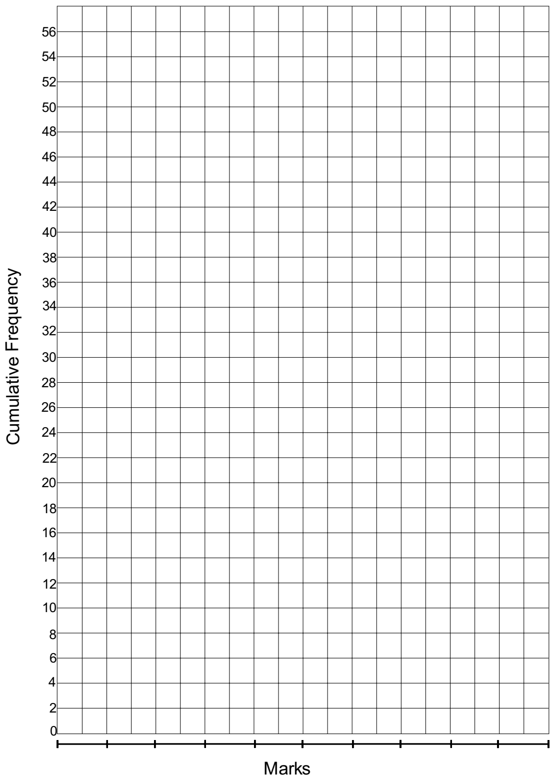
Marks	Frequency	Cumulative Frequency
Total		

(b) On the set of axes provided on the next page, draw a cumulative frequency graph (ogive curve) for this data.

(c) Determine the lower quartile.

(d) Determine the median.

(e) Determine the upper quartile.



Lesson

Cumulative Frequency Graphs (grouped data)

Determining **cumulative frequencies** is an effective way of representing **grouped data**. If you want to find the median of grouped data from a frequency table, a useful way to do this is by first determining the cumulative frequencies from the frequency table and then representing the information on a **cumulative frequency graph (or ogive curve)**. The method is illustrated below.

Example 2

A health and fitness group conducted a survey in the area to find out which age group most frequently makes use of gyms. The group determined the ages of their current clients. Gym contracts were then marketed to this age group in the area. The ages of current clients were recorded and then sorted.

Consider the frequency table of the Health and Fitness group. The data can be represented in a cumulative frequency table.

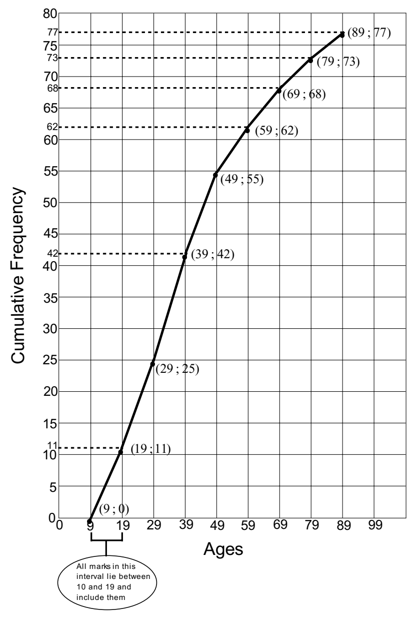
Class interval	Frequency	Cumulative Frequencies	
0 – 9 years	0	0	
10 – 19 years	11	0 + 11 = 11	11 people were 19 years or less
20 – 29 years	14	11 + 14 = 25	25 people were 29 years or less
30 – 39 years	17	25 + 17 = 42	42 people were 39 years or less
40 – 49 years	13	42 + 13 = 55	55 people were 49 years or less
50 – 59 years	7	55 + 7 = 62	62 people were 59 years or less
60 – 69 years	6	62 + 6 = 68	68 people were 69 years or less
70 – 79 years	5	68 + 5 = 73	73 people were 79 years or less
80 – 89 years	4	73 + 4 = 77	77 people were 89 years or less
90 – 99 years	0		
	Total: 77		

Note:

The total frequency of marks (77) is equal to the final cumulative frequency (77).

We can now represent the information graphically. The graph below is called a **cumulative frequency graph** or **ogive curve**.

The horizontal axis represents the age class intervals. The vertical axis represents the cumulative frequencies of the ages (0 –77).



Important deductions

1. The points on the graph have been joined by straight lines. If the number of points increases, the graph will take on the form of an S-shaped curve.
2. For each point, the first coordinate represents the **upper boundary** of the class interval. The second coordinate represents the cumulative frequency of the ages. For example, for the point (19; 11), there were 11 people whose ages were 19 years or less. The upper boundary here is 19.

3. We can use the graph to determine estimates of the quartiles for this data.

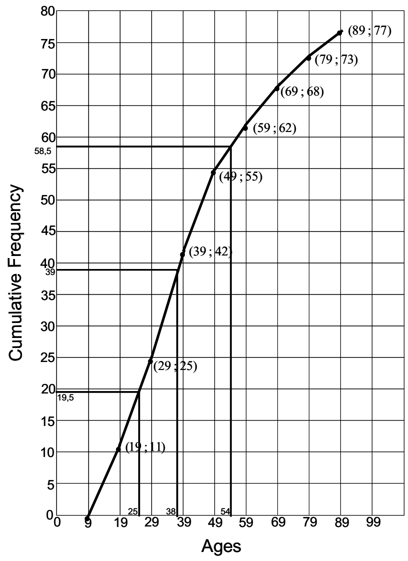
Position of $Q_1 = \frac{1}{4}(77 + 1) = 19{,}5$th postion (between the 19th and 20th age)

Reading off the age (on the *x*-axis) corresponding to the cumulative frequency 19,5 is approximately 25 years.

Position of $Q_2 = \frac{1}{2}(77 + 1) = 39$th age (median)

Reading off the age corresponding to the cumulative frequency 39 is approximately 38 years.

Position of $Q_3 = \frac{3}{4}(77 + 1) = 58{,}5$th position (between the 58th and 59th age)



Reading off the age corresponding to the cumulative frequency 58,5 is approximately 54 years.

Activity 2

INDIVIDUAL

SUMMATIVE ASSESSMENT

Complete this exercise in this workbook.

1. The following table (grouped frequency distribution) shows the mark obtained by 220 learners in a Science examination.

Percentage	1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Frequency	2	6	11	22	39	59	45	20	11	5

(a) Complete the cumulative frequency table for this data.

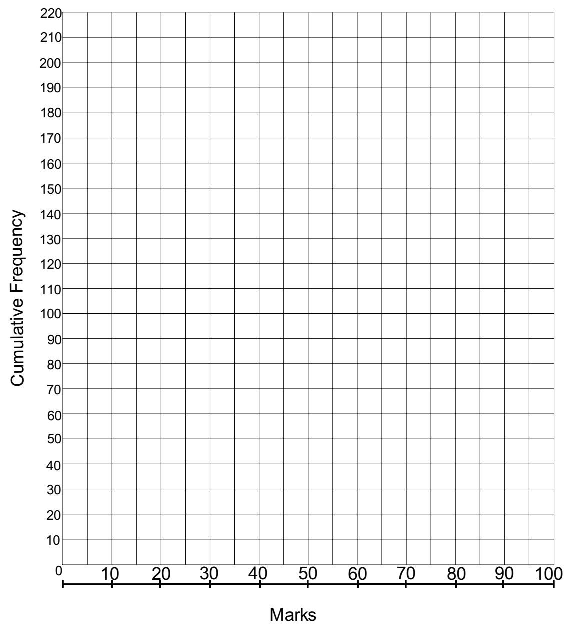
Marks	Frequency	Cumulative Frequency
1–10		
11–20		
21–30		
31–40		
41–50		
51–60		
61–70		
71–80		
81–90		
91–100		
Total		

(b) On the set of axes provided on the next page, draw a cumulative frequency graph (ogive curve) for this data.

(c) Determine the lower quartile.

(d) Determine the median.

(e) Determine the upper quartile.



2. The table represents the percentage of income spent on recreation by 50 people.

(a) Complete the cumulative frequency table for this data.

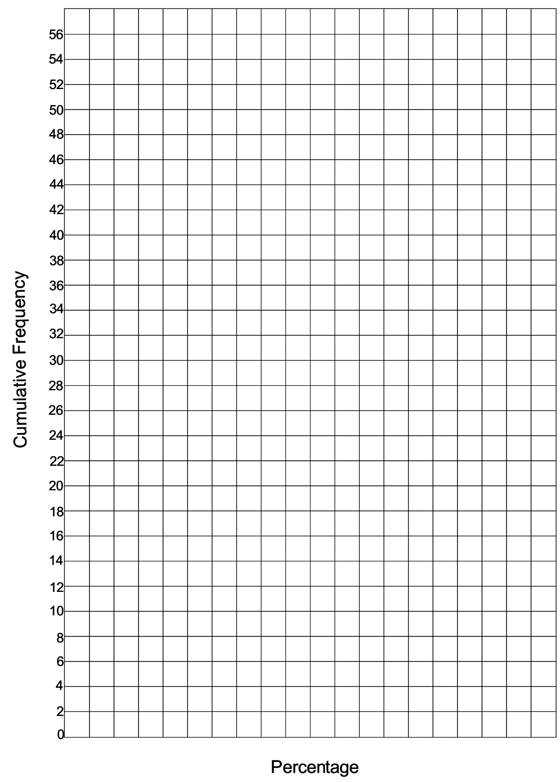
Percentage	Frequency	Cumulative Frequency
$12 < p \leq 18$	8	
$18 < p \leq 24$	20	
$24 < p \leq 30$	12	
$30 < p \leq 36$	8	
$36 < p \leq 42$	2	
Total		

(b) On the set of axes provided on the next page, draw a cumulative frequency graph (ogive curve) for this data.

(c) Determine the lower quartile.

(d) Determine the median.

(e) Determine the upper quartile.



DATA HANDLING (3)

Lesson 49

Learning Outcomes and Assessment Standards

Learning Outcome 4: Data Handling and Probability
Assessment Standard
Calculate and represent measures of central tendency and dispersion in univariate numerical data by:
- Five number summary
- Box and whisker diagrams
- Ogives
- Variance and standard deviation

Overview

In this lesson you will learn about:

- Variance.
- Standard deviation.

Lesson

LC DVD

Measures of dispersion (or spread) about the mean (ungrouped data)

There are two measures of dispersion about the mean:

- **Variance (s^2)**
- **Standard Deviation** (s or σ)

These measures of dispersion about the mean are highly effective since they use all of the data values. The range is not a good measure of dispersion because it doesn't eliminate outliers, which are not normally representative of the rest of the data. The Interquartile Range is better in the sense that it focuses on the central half of the data. The problem with the Interquartile Range is that it is centred about the median and uses only a few of the data values. The mean is a far better measure of central tendency and the variance and standard deviation are measures of dispersion about the mean. These measures are far better than the others.

We will now illustrate these two measures of dispersion by considering an example.

Example 1 (without using a calculator)

A basketball team consists of 10 players. The number of points each player scored during the season are as follows:

22 33 38 39 41 52 55 61 67 75

Step 1 Calculate the **mean**.

$$\bar{x} = \frac{\Sigma x}{n} = \frac{22+33+38+39+41+52+55+61+67+75}{10} = \frac{483}{10} = 48{,}3$$

Step 2 Calculate the **individual deviations from the mean**. Record your results in a table.

POINTS SCORED (x)	$x - \bar{x}$
22	22 – 48,3 = –26,3
33	33 – 48,3 = –15,3
38	38 – 48,3 = –10,3
39	39 – 48,3 = –9,3
41	41 – 48,3 = –7,3
52	52 – 48,3 = –3,7
55	55 – 48,3 = 6,7
61	61 – 48,3 = 12,7
67	67 – 48,3 = 18,7
75	75 – 48,3 = 26,7
	$\sum(x - \bar{x}) = 0$

You will probably notice that the sum of all these deviations from the mean equals 0, which is not very helpful if we are to make conclusions about how the data is spread about the mean.

Step 3 Calculate the **individual squared deviations from the mean**. Record your results in a table. Add these squared deviations.

By squaring the deviations, we are able to eliminate the negative signs so that we have more useful values.

POINTS SCORED (x)	$(x - \bar{x})$	$(x - \bar{x})^2$
22	22 – 48,3 = –26,3	691,69
33	33 – 48,3 = –15,3	234,09
38	38 – 48,3 = –10,3	106,09
39	39 – 48,3 = –9,3	86,49
41	41 – 48,3 = –7,3	53,29
52	52 – 48,3 = –3,7	13,69
55	55 – 48,3 = 6,7	44,89
61	61 – 48,3 = 12,7	161,29
67	67 – 48,3 = 18,7	349,69
75	75 – 48,3 = 26,7	712,89
	$\sum(x - \bar{x}) = 0$	$\sum(x - \bar{x})^2 = 2454{,}1$

Step 4 Determine the **variance** (mean of the squared deviations)

Formula 1 (Grade 11 and 12 Paper 2)

If you are working with an **entire population**, use:

$s^2 = \dfrac{\sum(x - \bar{x})^2}{n}$ to calculate the variance.

Formula 2 (Grade 12 Paper 3)

If you are working with a **random sample of a population**, use:

$s^2 = \dfrac{\sum(x - x)^2}{n - 1}$ to calculate the estimated variance of the population.

In this example, we are working with a basketball team. Therefore, we will use Formula 1 to calculate the variance because we are not dealing with a random sample of a larger population. The basketball team is an entire population in its own right.

variance(s^2) = $\frac{\sum(x-\bar{x})^2}{n} = \frac{2454,1}{10} = 245,41$

Step 5 Determine the **standard deviation (*s*)** by square rooting the variance.

standard deviation = s = $\sqrt{\frac{\sum(x-x)^2}{n}} = \sqrt{245,41} = 15,7$

Step 6 Determine the **standard deviation interval** for the data.

This interval is given by: $(\bar{x} - s; \bar{x} + s)$

For the example used, the standard deviation interval is:

(48,3 – 15,7; 48,3 + 15,7)

= (32,6 ; 64)

Step 7 Make **conclusions** about the spread of the data about the mean by establishing how many of the data values lie within or outside of the standard deviation interval.

In this example, it is clear that 7 of the 10 points lie within the standard deviation interval (33, 38, 39, 41, 52, 55 and 61). This means that most of the players performed well by scoring close to the mean points score. The teamwork was very good for this team.

Note: A small standard deviation indicates that the data items are clustered around the mean. A large standard deviation indicates that the items are more spread out.

Example 1 (by using a calculator)

The standard deviation can be easily calculated by using a calculator. We will use the CASIO *fx-82ES* to illustrate the method. The sequence is as follows:

Push the button: **MODE**

Push the button: **2 : STAT**

Push the button: **1 : 1 – VAR**

Enter the points: 22 = **33 = 38 = 39 = 41 = 52 = 55 = 61 =**
67 = 75 =

Push the button: **SHIFT**

Push the button: **STAT**

Push the button: **5: VAR**

Push the button: 3 **: *xσn***

Push the button: =

The answer will read: **15,7**

To clear calculator:

Push the buttons: **MODE**

Push the buttons: **1: COMP**

Example 2

Consider the following table which shows the number of learners who obtained certain marks out of 30 for a class test.

Marks (x)	18	19	20	21	22	23	24	25	26	27
No of learners	2	2	3	4	6	9	12	6	5	3

Calculate the mean and standard deviation for this data.

Solution

Marks x	**Freq f**	**f × x**	**$x - \bar{x}$**	**$(x - \bar{x})^2$**	**f × $(x - \bar{x})^2$**
18	2	36	–5,2	27,04	54,08
19	2	38	–4,2	17,64	35,28
20	3	60	–3,2	10,24	30,72
21	4	84	–2,2	4,84	19,36
22	6	132	–1,2	1,44	8,64
23	9	207	–0,2	0,04	0,36
24	12	288	0,8	0,64	7,68
25	6	150	1,8	3,24	19,44
26	5	130	2,8	7,84	39,2
27	3	81	3,8	14,44	43,32
Total	52	$\bar{x} = \frac{1206}{52} = 23{,}2$			258,08

The mean for this data is 23,2.

$s^2 = \frac{\sum f(x - \bar{x})^2}{n} = \frac{258{,}08}{52} = 4{,}963076923$ (variance)

The standard deviation is:

$s = \frac{\sum f(x - \bar{x})^2}{n} = \sqrt{4{,}963076923} = 2{,}23$

Example 2 (by using a calculator)

The standard deviation can be easily calculated by using a calculator. We will use the CASIO *fx-82ES* to illustrate the method. The sequence is as follows:

Push the button: **MODE**

Push the button: **2 : STAT**

Push the button: **1 : 1 – VAR**

Push the buttons: **SHIFT SETUP**

Scroll down and push:**3: STAT**

Push the button: **1: ON**

Enter the marks: 18 = 19 = 20 = 21 =

22 = 23 = 24 = 25 =

26 = 27 =

Enter the frequencies: 2 = 2= 3 = 4 =

6 = 9 = 12 = 6 =

5 = 3 =

Push the button: **SHIFT 1**

Push the button: **5: VAR**

Push the button: 3 **: *xσn***

Push the button: =

The answer will read: **2,23**

To clear calculator:

Push the buttons: **SHIFT SETUP**

Scroll down and push: **3: STAT**

Push the button: **2: OFF**

Push the buttons: **MODE**

Push the buttons: **1: COMP**

Activity 1

INDIVIDUAL

SUMMATIVE ASSESSMENT

1. The number of points scored by four Formula One racing drivers over a number of races are given below:

A	1	1	1	2	6	6	8	8	8	8	10	10	10
B	1	2	6	8	8	8	8	8	8	10	10	10	–
C	1	1	2	2	4	4	6	6	8	8	10	–	–
D	2	2	2	4	4	6	6	8	8	10	10	10	–

   (a) Calculate the mean and standard deviation for each driver without using a calculator.

   (b) Now calculate the standard deviation for each driver by using your calculator.

   (c) Discuss the performance of each driver by referring to the standard deviation for each driver.

2. The following table contains the number of learners who obtained certain marks on a class test out of 30.

Marks	20	21	22	23	24	25	26	27	28	29
No of learners	3	3	4	5	7	10	13	5	4	2

Calculate the standard deviation for this data by means of drawing a frequency table.

Lesson

DVD

Measures of dispersion about the mean (grouped data)

Example 3 (without using a calculator)

Consider the example of the Health and Fitness group. Calculate the standard deviation for this set of data.

Calculate the estimated mean:

Class interval	**Frequency**	**Midpoint class interval**	**Frequency × Midpoint**
0 – 9 years	0	–	–
10 – 19 years	11	14,5	159,5
20 – 29 years	14	24,5	343
30 – 39 years	17	34,5	586,5
40 – 49 years	13	44,5	578,5

50 – 59 years	7	54,5	381,5
60 – 69 years	6	64,5	387
70 – 79 years	5	74,5	372,5
80 – 89 years	4	84,5	338
90 – 99 years	0	–	–
Totals	77	–	3146,5

Estimated mean = $\frac{3146,5}{77}$ = 40,8

Calculate the squared deviations from the mean:

Class interval	**f**	**Midp (m)**	**f × m**	**m – $\bar{x}$**	**(m – $\bar{x}$)²**	**f·(m – $\bar{x}$)²**
0-9 years	0	–	–	–	–	–
10-19 years	11	14,5	159,5	14,5 – 40,8 = –26,3	691,69	7608,59
20-29 years	14	24,5	343	24,5 – 40,8 = –16,3	265,69	3719,66
30-39 years	17	34,5	586,5	34,5 – 40,8 = –6,3	39,69	674,73
40-49 years	13	44,5	578,5	44,5 – 40,8 = 3,7	13,69	177,97
50-59 years	7	54,5	381,5	54,5 – 40,8 = 13,7	187,69	1313,83
60-69 years	6	64,5	387	64,5 – 40,8 = 23,7	561,69	3370,14
70-79 years	5	74,5	372,5	74,5 – 40,8 = 33,7	1135,69	5678,45
80-89 years	4	84,5	338	84,5 – 40,8 = 43,7	1909,69	7638,76
90-99 years	0	–	–	–	–	–
Totals	77	–	3146,5			30182,13

Calculate the variance:

$$s^2 = \frac{\sum f\cdot(m-\bar{x})^2}{n} = \frac{30182,13}{77} = 391,9757143$$

Calculate the standard deviation:

$$s^2 = \sqrt{\frac{\sum f\cdot(m-\bar{x})^2}{n}} = \sqrt{391,9757143} = 19,8$$

Example 3 (by using a calculator)

The standard deviation can be easily calculated by using a calculator. We will use the CASIO *fx-82ES* to illustrate the method. The sequence is as follows:

Push the button: **MODE**

Push the button: **2 : STAT**

Push the button: **1 : 1 – VAR**

Push the buttons: **SHIFT SETUP**

Scroll down and push:**3: STAT**

Push the button: **1: ON**

Enter the midpoints: 14,5 = **24,5 = 34,5 = 44,5 = 54,5 = 64,5 = 74,5 = 84,5 =**

Enter the frequencies: 11 = 14 = 17 = 13 = 7 = 6 = 5 = 4 =

Push the button: **SHIFT 1**

Push the button: **5: VAR**

Push the button: **3 : *xσn***

Push the button: =

The answer will read: **19,8**

To clear calculator:

Push the buttons: **SHIFT SETUP**

Scroll down and push: **3: STAT**

Push the button: **2: OFF**

Push the buttons: **MODE**

Push the buttons: **1: COMP**

Activity 2

SUMMATIVE ASSESSMENT

INDIVIDUAL

1. The table represents the percentage of income spent on recreation by 50 people.

   (a) Complete the following table for this data.

Percentage	**f**	**Midp (m)**	**f × m**	**m – $\bar{x}$**	**(m – $\bar{x}$)^2**	**f × (m – $\bar{x}$)^2**
$12 < p \leq 18$	8					
$18 < p \leq 24$	20					
$24 < p \leq 30$	12					
$30 < p \leq 36$	8					
$36 < p \leq 42$	2					
Total			$\bar{x}$ =			

   (b) Calculate the mean for this data.

   (c) Now calculate the standard deviation for this data.

   (d) Now verify your answer by using a calculator.

Lesson 50

DATA HANDLING (4)

Mathematical modelling

Learning Outcomes and Assessment Standards

Learning Outcome 4: Data Handling and Probability
Assessment Standard
Represent bivariate numerical data as a scatter plot and suggest intuitively whether a linear, quadratic or exponential function would best fit the data (problems should include issues related to health, social, economic, cultural, political and environmental issues).

Overview

In this lesson you will learn about:

- Lines and curves of best fit.
- Correlation of data

LC DVD

Lesson

Bivariate data

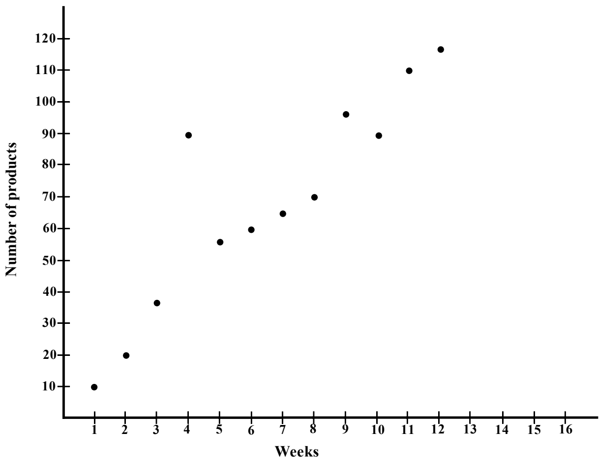
In the previous episodes on Data Handling, we worked with univariate data, which involved one variable only. For example, if you are working with the number of learners who achieved certain marks on a class test, you are working with univariate data, because the only variable involved is the marks. Working with bivariate data involves working with two variables. For example, if you are working with two variables such as time in seconds and cost in rands, you are working with bivariate data.

We can represent this data by means of what is called a **scatter plot**. If the data shows a trend (or pattern), we can try to identify whether the trend forms a linear function or some other type of function such as a quadratic or exponential function. A line or curve of best fit can be intuitively drawn on the scatter plot diagram and predictions can then be made.

Scatter plots and lines and curves of best fit

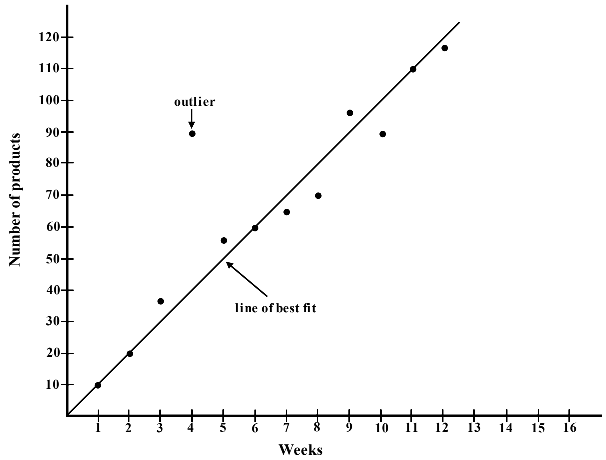
Lines of best fit

Consider the following scatter plot of information obtained by a company which recorded the number of products sold per week.



It is clear from the above graph that the points tend to form a pattern or trend which resembles a straight line. The points however, do not lie on a perfect straight line. We can draw what is called a **line of best fit** which will help us to predict future values. In order to do this, draw a line through some of the points with the aim of having the same number of points above the line as below the line.

The line of best fit will now be drawn onto the graph. Refer to the diagram which follows on the next page.



A line can be drawn through four of the points with four above and four below. One of the points (see diagram) is way out compared to the others. This point is called an **outlier**.

A line of best fit will not represent the data perfectly, but it will give you an idea of the trend. Different people will probably draw slightly different lines of best fit, but the trend should be the more or less the same in each case.

The equation of the line of best fit that we have drawn above can be obtained as follows:

Select two points on this line, calculate the gradient and hence the equation.

We can choose the points (2; 20) and (6; 60)

gradient $= \frac{60-20}{6-2} = \frac{40}{4} = 10$

y-intercept of line is 0

$\therefore$ The equation of the line of best fit is given by: $y = 10x$

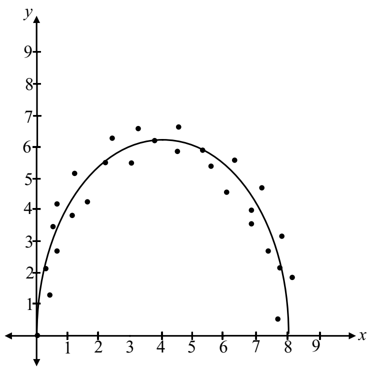
where x represents the number of weeks and y the number of products sold per week

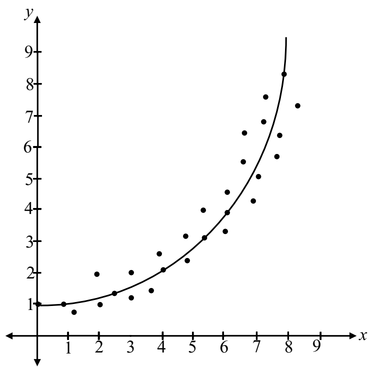
You can now predict the number of products sold in 16 weeks by using the equation of the line of best fit:

For $x = 16$ $\quad y = 10(16) = 160$ products (estimation)

Curves of best fit

It is sometimes the case that bivariate data can have curves of best fit rather than lines of best fit. The types of curves of best fit can take the form of a quadratic or exponential function. Consider, for example, the following scatter plots.



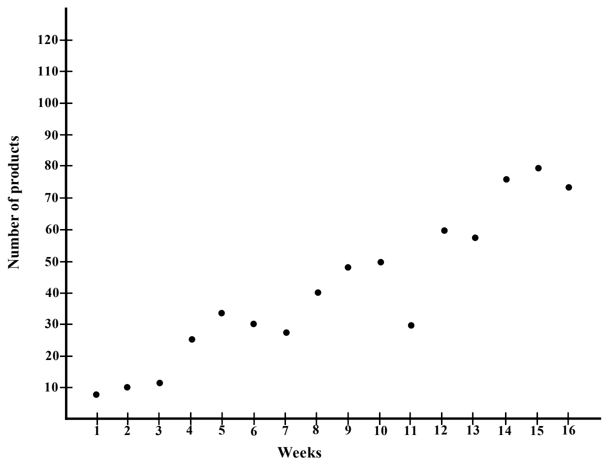


Activity 1

SUMMATIVE ASSESSMENT

INDIVIDUAL

1. A company records the number of products sold per week.



(a) Draw a line of best fit on the diagram above.

(b) Determine the equation of your line of best fit.

(c) Predict how many products will be sold after 40 weeks.

2. The table below represents the number of people infected with HIV in a certain region from the year 2002 to 2007.

Year	Number of people infected with HIV
2002	118
2003	123
2004	131
2005	134
2006	136
2007	138

(a) Draw a scatter plot to represent this data.

(b) Explain whether a linear, quadratic or exponential curve would be a line or curve of best fit. Hence draw this curve on your scatter plot.

(c) If the same trend continued, estimate, by using your graph, the number of people that will be infected in 2008.

Lesson

DESCRIBING TRENDS IN A SCATTER PLOT

Positive trend: If a line of best fit has a positive gradient, we say that the data has a **positive trend**. As the variable on the *x*-axis increases, the variable on the *y*-axis also increases.

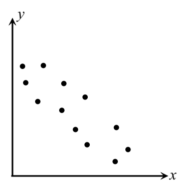
Negative trend: If a line of best fit has a negative gradient, we say that the data has a **negative trend**. As the variable on the *x*-axis increases, the variable on the *y*-axis decreases.

Correlation: If the points on the scatter plot are close to the line of best fit, we have a **strong correlation or association** between the two variables. If the points are not so close to the line of best fit, we have a **weak correlation or association** between the two variables.

Consider the following examples.

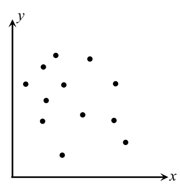
Here we have a strong positive correlation between the two variables.

Here we have a weak positive correlation between the two variables.



Here we have a strong negative correlation between the two variables.

Here we have a weak negative correlation between the two variables.



Here we do not have any correlation between the variables.

ANSWERS AND SOLUTIONS

Lesson 1

Solutions to Worksheet 1

1. $\frac{(x+2)(x-3)}{(3-x)6}$

$\frac{x+2}{-6}$

2. $\frac{9-12x+4x^2}{2x^2-x-3} \times \frac{4+x-3x^2}{6x^2-17x+12}$

$= \frac{(3-2x)^2}{(2x-3)^2(x+1)} \times \frac{(4-3x)(1+x)}{(3x-4)}$

$= \frac{\cancel{(3-2x)^2}}{\cancel{(2x-3)^2}\cancel{(x+1)}} \times \frac{^{(-1)}\cancel{(4-3x)}\cancel{(1+x)}}{\cancel{(3x-4)}}$

$= -1$

3. $\frac{(x-1)-2(1-x)^2}{(2x-3)(x+2)}$

$= \frac{(x-1)[1-2(x-1)]}{(2x-3)(x+2)}$

$= \frac{(x-1)\cancel{(3-2x)}^{(-1)}}{\cancel{(2x-3)}(x+2)}$

$= \frac{-x+1}{x+2}$

4. $\frac{(p-5)(p+3)}{6(p-2)} \times \frac{2-p}{5-p}$

$= \frac{p+3}{6}$

5. $\frac{y(x-3)+4(x-3)}{(3-x)} \times \frac{2y}{y(4+y)}$

$= \frac{^{(-1)}\cancel{(x-3)}\cancel{(y+4)}}{\cancel{(3-x)}} \times \frac{2y}{y\cancel{(4+y)}}$

$= -2$

Solutions to Worksheet 2

1. $\frac{1}{y-2} - \frac{3}{y-3} - \frac{4}{(y-3)^2}$

$= \frac{(y-3)^2 - 3(y-2)(y-3) - 4(y-2)}{(y-2)(y-3)^2}$

$= \frac{y^2-6y+9-3(y^2-5y+6)-4y-8}{(y-2)(y-3)^2}$

$= \frac{-2y^2+5y-17}{(y-2)(y-3)^2}$

2. $\frac{3x}{x+2} + \frac{2}{2x-5} + \frac{4x}{2}$

$= \frac{6x(2x-5)+4(x+2)+4x(x+2)(2x-5)}{2(x+2)(2x-5)}$

$= \frac{12x^2-30x+4x+8+4x(2x^2-x-10)}{2(x+2)(2x-5)}$

$= \frac{8x^3+8x^2-66x+8}{2(x+2)(2x-5)}$

3. $\frac{-6}{(x-3)(x+3)} - \frac{3}{(x+3)^2} + \frac{1}{x-3}$

$= \frac{-6(x+3)-3(x-3)+(x+3)^2}{(x-3)(x+3)^2}$

$= \frac{-6x-18-3x+9+x^2+6x+9}{(x-3)(x+3)^2}$

$= \frac{x^2-3x}{(x-3)(x+3)^2}$

4. $\frac{3y+2x}{y+x} - \frac{3}{(3y+2x)(y+x)}$

$= \frac{(3y+2x)^2-3}{(y+x)(3y+2x)}$

$= \frac{9y^2+12xy+4x^2-3}{(y+x)(3y+x)}$

5. $x + \frac{1}{x-1} - \frac{2}{x-2}$

$= \frac{x(x-1)(x-2)+(x-2)-2(x-1)}{(x-1)(x-2)}$

$= \frac{x^3-3x^2+2x+x-2-2x+2}{(x-1)(x-2)}$

$= \frac{x^3-3x^2+x}{(x-1)(x-2)}$

6. $\frac{2}{(m-1)^2} - \frac{3}{m-1}$

$= \frac{2-3(m-1)}{(m-1)^2}$

$= \frac{2-3m+3}{(m-1)^2}$

$= \frac{5-3m}{(m-1)^2}$

Solutions to Worksheet 3

1) $\frac{2}{2x-3}+\frac{4}{x}$

$=\frac{2x+8-12}{x(2x-3)}$

$=\frac{10x-12}{x(2x-3)}$

2. $\frac{2}{2x-3}=\frac{4}{x}$; $\quad 2x-3\neq 0 \Rightarrow x\neq \frac{3}{2}$ and $x\neq 0$

$\therefore \frac{2}{2x-3}\times x(2x-3)=\frac{4}{x}\times x(2x-3)$

$\therefore 2x=4(2x-3)$

$\therefore 2x=8x-12$

$\therefore -6x=-12$

$\therefore x=2$

3. $2+\frac{5}{x-1}$

$\therefore \frac{2(x-1)+5}{x-1}$

$\therefore \frac{2x-2+5}{x-1}$

$\therefore \frac{2x+3}{x-1}$

4. $2=\frac{5}{x-1}$; $x-1\neq 0$; $x\neq 1$

$\therefore 2(x-1)=5$

$\therefore 2x-2=5$

$\therefore 2x=7$

$\therefore x=\frac{7}{2}$

5. $\frac{3}{2x}+\frac{4}{3x}-\frac{1}{x+3}$

$=\frac{9(x+3)+8(x+3)-6x}{6x(x+3)}$

$=\frac{9x+27\ 8x+24-6x}{6x(x+3)}$

$=\frac{11x+51}{6x(x+3)}$

6. $\frac{3}{2x}+\frac{4}{3x}=\frac{1}{x+3}$

Restrictions on x: $2x\neq 0 \quad 3x\neq 0\ x+3\neq 0$
$x\neq 0 \quad x\neq 0 \quad \therefore x\neq -3$

$\therefore \frac{3}{2x}\times 6x(x+3)+\frac{4}{3x}\times 6x(x+3)$

$=\frac{1}{x+3}\times 6x(x+3)$

$\therefore 9(x+3)+8(x+3)=6x$

$\therefore 9x+27+8x+24=6x$

$\therefore 11x=-51$

$\therefore x=-\frac{51}{11}$

7. $\frac{2}{x+1}+\frac{3}{1-x}$

$\therefore \frac{2(x-1)-3(x+1)}{(x+1)(x-1)}$

$\therefore \frac{2x-2-3x-3}{(x+1)(x-1)}$

$\therefore \frac{-x-5}{(x+1)(x-1)}$

8. $\frac{2}{x+1}+\frac{3}{1-x}=0$

Restrictions on x: $x+1\neq 0$; $\quad x\neq -1$
and $\quad x-1\neq 0$; $x\neq 1$

$\therefore \frac{2}{x+1}-\frac{3}{x-1}=0$

$\therefore \frac{2}{x+1}=\frac{3}{x-1}$

$\therefore 2(x-1)=3(x+1)$

$\therefore 2x-2=3x+3$

$\therefore -x=5$

$\therefore x=-5$

Lesson 2

Solutions to Activity 1

1. $(4)^2 = 16$
2. $\left(\frac{7}{2}\right)^2 = \frac{49}{4}$
3. $(-3)2 = 9$
4. $\left(-\frac{5}{2}\right)^2 = \frac{25}{4}$
5. $\left(\frac{1}{4} \times 2\right)^2 = \frac{1}{4}$
6. $\left(-\frac{2}{5} \times 2\right)^2 = \frac{16}{25}$

Solutions to Activity 2

1. a) $x^2 + 3x - 7$

$= x^2 + 3x + \frac{9}{4} - \frac{9}{4} - 7$

$= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{28}{4}$

$= \left(x + \frac{3}{2}\right)^2 - \frac{37}{4}$

b) $-\left[x^2 + 5x + \left(\frac{5}{2}\right)^2 - \frac{25}{4} + 1\right]$

$= -\left(x + \frac{5}{2}\right)^2 + \frac{25}{4} - \frac{4}{4}$

$= -\left(x + \frac{5}{2}\right)^2 + \frac{21}{4}$

Max. value of $\frac{21}{4}$ at $x = -\frac{5}{2}$

c) $3x^2 + 2x + 7$

$= 3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9} + \frac{7}{3}\right)$

$= 3\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{21}{9}\right]$

$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{20}{9}\right]$

$= 3\left(x + \frac{1}{3}\right)^2 + \frac{20}{3}$

Min value of $\frac{20}{3}$ when $x = -\frac{1}{3}$

d) $\left[x^2 + \frac{4}{5} + \left(\frac{2}{5}\right)^2 - \frac{4}{25} + \frac{3}{5}\right]$

$= 5\left(x + \frac{2}{5}\right)^2 - \frac{4}{5} + 3$

$= 5\left(x + \frac{2}{5}\right)^2 + \frac{11}{5}$

Min. value $\frac{11}{5}$ at $x = -\frac{2}{5}$

e) $x^2 + px + \left(\frac{p}{2}\right)^2 - \frac{p^2}{4} = 3$

$= \left(x + \frac{p}{2}\right)^2 + 3 - \frac{p^2}{4}$

$= \left(x + \frac{p}{2}\right)^2 + \frac{12 - p^2}{4}$

Min. value of $\frac{12 - p^2}{4}$ at $x = \frac{-p}{2}$

f) $px^2 + qx + r$

$= p\left(x^2 + \frac{q}{p}x + \frac{q^2}{4p^2} - \frac{q^2}{4p^2} + \frac{r}{p}\right)$

$= p\left[\left(x + \frac{q}{2p}\right)^2 - \frac{q^2}{4p^2} + \frac{4pr}{4p^2}\right]$

$= p\left(x + \frac{q}{2p}\right)^2 + \frac{4pr - q^2}{4p}$

If $p < 0$; max value is $\frac{4pr - q^2}{4p}$ when $x = -\frac{q}{2p}$

If $p > 0$; min value $\frac{4pr - q^2}{4p}$ when $x = -\frac{q}{2p}$

2. a) $x^2 - x + \left(-\frac{1}{2}\right)^2 - \frac{1}{4} + 5$

$= \left(x - \frac{1}{2}\right)^2 + \frac{19}{4}$

Min. value is $+\frac{19}{4}$

$\therefore$ the expression is always positive.

b) $2[x^2 - \frac{3}{2}x + \left(\frac{-3}{4}\right)^2 - \frac{9}{16} + 4]$

$= 2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8} + 8$

$= 2\left(x - \frac{3}{4}\right)^2 + \frac{55}{8}$

Min. value is $\frac{55}{8}$

$\therefore$ the expression is always positive.

c) $10[x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \frac{1}{16} + \frac{1}{5}$

$= 10\left(x + \frac{1}{4}\right)^2 - \frac{5}{8} + 2$

$= 10\left(x + \frac{1}{4}\right)^2 + \frac{11}{8}$

$= 10\left(x + \frac{1}{4}\right)^2 \geq 0$

$\therefore\ 10\left(x + \frac{1}{4}\right)^2 + \frac{11}{8} \geq \frac{11}{8} > 0$

3. a) $-[x^2 - 3x + 5]$

$= -\ [x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \frac{9}{4} + 5]$

$= -\left(x - \frac{3}{2}\right)^2 + \frac{9}{4} - 5$

$= -\left(x - \frac{3}{2}\right)^2 - \frac{11}{4}$

Max. value of $-\frac{11}{4}$

$\therefore$ the expression is always negative.

b) $-4(x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 - \frac{1}{16} + \frac{1}{4})$

$= -4\left(x - \frac{1}{4}\right)^2 + \frac{1}{4} - 1$

$= -4\left(x - \frac{1}{4}\right)^2 - \frac{3}{4}$

$\left(x - \frac{1}{4}\right)^2 \geq 0$

$\therefore -4\left(x - \frac{1}{4}\right)^2 \leq 0$

$\therefore -4\left(x - \frac{1}{4}\right)^2 - \frac{3}{4} \leq -\frac{3}{4} < 0$

4. i) $A = x(40 - x)$

$A = -x^2 + 40x$

ii) $A = -(x^2 - 40x + (-20)^2 - 400)$

$a = -(x - 20)^2 + 400$

Max. is 400 when $x = 20$

Max. area is 400 m^2

iii) dimension $x = 20$

$\therefore L = 20 \quad B = 20$

Lesson 3

Activity 1

1. $\left[\frac{3^3a^4}{2^3a}\right]^{\frac{2}{3}}$

$= \left[\left(\frac{3a}{2}\right)^3\right]^{\frac{2}{3}}$

$= \frac{9a^2}{4}$

2. $\left[\frac{2^4y^6}{3^2x^4}\right]^{-\frac{1}{2}}$

$= \left[\frac{3^2x^4}{2^4y^6}\right]^{\frac{1}{2}}$

$= \frac{3x^2}{4y^3}$

3. $\frac{(2.5)^{n+3}.5^{n-1}}{(5^2.2)^{n+2}}$

$= \frac{2^{n+3}.5^{n+3}.5^{n-1}}{5^{2n+4}.2^{n+2}}$

$= 2^{n+3-n-2}.5^{2n+2-2n-4}$

$= 2.5^{-2}$

$= \frac{2}{25}$

4. $\frac{(3^2)^{2k+1}.3^{-k}.(2^2)^{2-k}}{3^{5k-2}.(2.3)^{3-2k}.2}$

$= \frac{3^{4k+2}.3^{-k}.2^{4-2k}}{3^{5k-2}.2^{3-2k}.3^{3-2k}.2}$

$= 3^{4k+2-k-5k+2-3+2k}.2^{4-2k-3+2k-1}$

$= 3^1.2^0$

$= 3$

5. $\frac{3^{2n-1}.3^{-8n}.3^{3n+3}}{3^{-3n-1}}$

$= 3^{2n-1-8n+3n+3+3n+1}$

$= 3^3$

$= 27$

6. $\frac{3^x(2^1 3^1)^{2x-1}\cdot 2^{4x+1}}{(2^2\cdot 3)^{3x}}$

$= \frac{3^x\cdot 2^{2x-1}\cdot 3^{2x-1}\cdot 2^{4x+1}}{2^{6x}.3^{3x}}$

$= 3^{x+2x-1-3x}\cdot 2^{6x-6x} = 3^{-1}\cdot 2^0 = \frac{1}{3}$

Activity 2

1. $\frac{3^x + 3^x\cdot 3}{3^x - 3^x\cdot 3^2}$

$= \frac{3^x(1+3)}{3^x(1-9)}$

$= -\frac{4}{8}$

$= -\frac{1}{2}$

2. $\frac{2\cdot 2^x\cdot 2^1 + 8\cdot 2^x\cdot 2^{-3}}{4\cdot 2^x\cdot 2^{-1} - 16\cdot 2^x\cdot 2^{-4}}$

$= \frac{\cancel{2^x}\left(8 + 8 \times \frac{1}{8}\right)}{\cancel{2^x}\left(2 - 16 \times \frac{1}{16}\right)}$

$= \frac{9}{1}$

$= 9$

3. $\frac{6\cdot 3^x\cdot 3^{-1} - 2\cdot 3^x\cdot 3^1}{5\cdot 3^x + 3^x\cdot 3^2}$

$= \frac{3^x(2-6)}{3^x(5+9)}$

$= -\frac{4}{14} = -\frac{2}{7}$

4. $\dfrac{5^x + 5^x.5^{-2}}{2.5^x.5^{-1} - 3.5^x.5^{-2}}$

$= \dfrac{5^x\left(1 - \frac{1}{25}\right)}{5^x\left(\frac{2}{5} - \frac{3}{25}\right)}$

$= \dfrac{24}{25} \div \dfrac{10-3}{25}$

$= \dfrac{24}{25} \times \dfrac{25}{7}$

$= \dfrac{24}{7}$

5. $(a + b) - \left(a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b\right)$

$= a + b - a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} - b$

$= 2a^{\frac{1}{2}}b^{\frac{1}{2}}$

Lesson 4

Activity 1

1. $x = 3$ or $x = -7$

2. $x = 5$ or $x = 8$

3. $x = \frac{3}{2}$ or $x = \frac{4}{3}$

4. $x = 5$ (note $x \neq -1$)

5. Restr: $x \neq 0$; $x \neq 2$

LCD: $2x(x - 2)$

$\therefore\ 60x - x(x - 2) = 30(x - 2).2$

$\therefore\ 60x - x^2 + 2x = 60x - 120$

$\therefore\ x^2 - 2x - 120 = 0$

$\therefore\ (x + 10)(x - 12) = 0$

$\therefore\ x = -10$ or $x = 12$

6. $\frac{4x}{3(x+4)} - \frac{1}{2} = \frac{1}{2(x-1)}$ Restr: $x \neq -4$; $x = 1$

LCD: $6(x - 1)(x + 4)$

$\therefore\ 2.4x(x - 1) - 3(x - 1)(x + 4) = 3(x + 4)$

$\therefore\ 8x^2 - 8x - 3(x^2 + 3x - 4) = 3x + 12$

$\therefore\ 5x^2 - 17x + 12 = 3x + 12$

$\therefore\ 5x^2 - 20x = 0$

$\therefore\ x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x = 4$

7. Restr: $x \neq 6$; $x \neq -2$

LCD: $8(x + 2)(x - 6)$

$21(x + 2) - 5(x - 6) + 8(x + 2)(x - 6) = 0$

$\therefore\ 21x + 42 - 5x + 30 + 8(x^2 - 4x - 12) = 0$

$\therefore\ 16x + 72 + 8x^2 - 32x - 96 = 0$

$\therefore\ 8x^2 - 16x - 24 = 0$

$\therefore\ x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

$x = -1$ or $x = 3$

8. LCD: $3x(x + 1)$

Restr: $x \neq 0$; $x \neq -1$

$\frac{x+1}{x} - \frac{5x}{3(x+1)} = \frac{2}{3}$

$\therefore\ 3(x + 1)^2 - 5x^2 = 2x(x + 1)$

$\therefore\ 3(x^2 + 2x + 1) - 5x^2 = 2x^2 + 2x$

$\therefore\ 3x^2 + 6x + 3 - 5x^2 = 2x^2 + 2x$

$\therefore\ 4x^2 - 4x - 3 = 0$

$(2x + 1)(x - 3) = 0$

$x = -\frac{1}{2}$ or $x = 3$

9. $x = \frac{3}{5}$ or $x = 3$

10. a) $x = 0$

b) $x = 0$ or $x = -\frac{3}{2}$

c) $x = \pm\sqrt{3}$

d) $x = 0$ or $x = -\frac{3}{2}$ or $x = \pm\sqrt{3}$

11. $\dfrac{2+\frac{1}{x}}{2} = \dfrac{3-\frac{1}{x}}{3+\frac{1}{x}}$

$\dfrac{2x+1}{2x} = \dfrac{3x-1}{3x+1}$

Restr: $x \neq 0 \quad x \neq -\frac{1}{3}$

LCD: $2x(3x + 1)$

$\Rightarrow \quad (2x + 1)(3x + 1) = 2x(3x - 1)$

$6x^2 + 5x + 1 = 6x^2 - 2x$

$\therefore\ 7x = -1$

$x = -\frac{1}{7}$

Activity 2

1. $2x + 1 + 2x - 1 = 4x$ which is an even number

$= 2(2x)$

2. $x^2 - (x - 1)^2$ which is an odd number

$= x^2 - (x^2 - 2x + 1)$

$= x^2 - x^2 + 2x - 1$

$= 2x - 1$

3. a) $7^2 - 4^2$ this is a multiple of three

$= 49 - 16$

$= 33$

$= 3(11)$

b) $100 - 36$ this is a multiple of 4

$= 64$

$= 4(16)$

c) $144 - 49$ this is a multiple of 5

$= 5(19)$

d) The square of any number minus the square of another number is a multiple of the difference between the two numbers.

e) Let the one number be n and the other number be m with $n > m$.

So: $n^2 - m^2 = (n - m)(n + m)$

Thus with $(n - m)$ being a factor of $n^2 - m^2$, it means that $n^2 - m^2$ is a multiple of $(n - m)$.

this means that the answer is divisible by $(n - m)$

Question 4 is for you to do on your own.

Lesson 5

Solutions to Activity 1

1. (a) $x = 5$

b) $\sqrt{11 - x} = x + 1$

Restr: $11 - x \geq 0$ and $x + 1 \geq 0$

$x \leq 1$ $\quad x \geq -1$

$\therefore -1 \leq x \leq 1$

Now $11 - x = x^2 + 2x + 1$

$x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

$\therefore x = -5$ or $x = 2$

But $-1 \leq x \leq 1$

$\therefore$ No solution

c) $\sqrt{5x - 1} = 2x - 1$

Restr: $5x - 1 \geq 0$ and $2x - 1 \geq 0$

$x \geq \frac{1}{5}$ and $x \geq \frac{1}{2}$

$\therefore x \geq \frac{1}{2}$

Now: $5x - 1 = 4x^2 - 4x + 1$

$\therefore 4x^2 - 9x + 2 = 0$

$(4x - 1)(x - 2) = 0$

$\therefore x = \frac{1}{4}$ or $x = 2$

d) $\sqrt{x - 2} = 4 - x$

Restr: $x - 2 \geq 0$ or $4 - x \geq 0$

$x \geq 2$ or $x \leq 4$

$\therefore 2 \leq x \leq 4$

$x - 2 = x^2 - 8x + 16$

$\therefore x^2 - 9x + 18 = 0$

$(x - 3)(x - 6) = 0$

$x = 3$ or $x = 6$

e) $\sqrt{10 - 3x} = x - 2$

Restr: $10 - 3x \geq 0$ or $x - 2 \geq 0$

$3x \leq 10$ $\quad x \geq 2$

$x \leq \frac{10}{3}$

$\therefore x \leq x \leq \frac{10}{3}$

$\therefore 10 - 3x = x^2 - 4x + 4$

$\therefore x^2 - x - 6 = 0$

$(x + 2)(x - 3) = 0$

$\therefore x = -2$ or $x = 3$

f) $(x - 3)\sqrt{x - 2} - 5(x - 3) = 0$

$\therefore (x - 3)\left(\sqrt{x - 2} - 5\right) = 0$ Restr: $x - 2 \geq 0$

$\therefore x = 3$ or $\sqrt{x - 2} = 5$ $\quad \therefore x \geq 2$

$x - 2 = 25$

$x = 27$

2. (b) : $x = 2$ only (c) $x \leq -2$; So $x = -14$

3. $x = 41$

4. $x = \frac{9}{11}$ or $x = -\frac{3}{2}$

Solution to Activity 2

2. $n \geq 0 \quad m \geq 0$

3. $c \leq 0 \qquad a \leq 0$

Lesson 6

1. $x^2 - 2x + (1)^2 - 1 - 7 = 0$

   $(x - 1)^2 = 8$

   $x - 1 = \pm\sqrt{8}$

   $x = 1 \pm 2\sqrt{2}$

   $4x^2 - 8x + 4 = 28 + 4$

   $(2x - 2)^2 = 32$

   $2x - 2 = \pm 4\sqrt{2}$

   $x - 1 = \pm 2\sqrt{2}$

   $x = 1 \pm 2\sqrt{2}$

2. $x^2 - 4x + 7 = 0$

   $\therefore x^2 - 4x + 4 + 3 = 0$

   $\therefore (x - 2)^2 = -3$

   x is non $\mathbb{R}$.

3. $p^2 - 3p + \left(-\frac{3}{2}\right)^2 - \frac{9}{4} + 1 = 0$

   $\left(p - \frac{3}{2}\right)^2 = \frac{9}{4} - 1$

   $p - \frac{3}{2} = \pm\sqrt{\frac{9-4}{4}}$

   $p = \frac{+3}{2} \pm \frac{\sqrt{5}}{2}$

   $p = \frac{+3 \pm \sqrt{5}}{2}$

   $4p^2 - 12p + 9 = -4 + 9$

   $(2p - 3)^2 = 5$

   $2p - 3 = \pm\sqrt{5}$

   $p = \frac{3 \pm \sqrt{5}}{2}$

4. $2x^2 - 3x - 1 = 0$

   $4a = 8$ $16x^2 - 24x + 9 = 8 + 9$

   $b^2 = 9$ $(4x - 3)^2 = 17$

   $\therefore 4x - 3 = \pm\sqrt{17}$

   $x = \frac{3 \pm \sqrt{17}}{4}$

5. $x^2 - 2x + 1 = 24 + 1$

   $(x - 1)^2 = 25$

   $\therefore x - 1 = \pm 5$

   $x = 1 \pm 5$

   $\therefore x = 6$ or $x = -4$

6. $5x^2 - 3x - 1 = 0$

   $4a = 20 \qquad 100x^2 - 60x + 9 = +20 + 9$

   $b^2 = 9 \qquad \therefore (10x - 3)^2 = 29$

   $\therefore 10x - 3 = \pm\sqrt{29}$

   $\therefore x = \frac{3 \pm \sqrt{29}}{10}$

7. $-2x^2 - x + 5 = 0$

$2x^2 + x - 5 = 0$

$x^2 + \frac{1}{2}x \qquad -\frac{5}{2} = 0$

$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5}{2} = 0$

$\left(x + \frac{1}{4}\right)^2 = \frac{1}{16} + \frac{5}{2}$

$x + \frac{1}{4} = \pm\sqrt{\frac{1 + 40}{16}}$

$x = -\frac{1}{4} \pm \frac{\sqrt{41}}{4}$

$= \frac{-1 \pm \sqrt{41}}{4}$

$4a = -8$

$b^2 = 1$

$16x^2 + 8x + 1 = 40 + 1$

$(4x + 1)^2 = 41$

$x = \frac{-1 \pm \sqrt{41}}{4}$

8. $3x^2 - 6x - 2 = 0$

$4a = 12 \qquad 36x^2 - 72x + 36 = 24 + 36$

$b^2 = 36 \qquad (6x - 6)^2 = 60$

$\therefore 6x - 6 = \pm\sqrt{60}$

$\therefore x = \frac{6 \pm \sqrt{60}}{6}$

9. $9x^2 - 6mx + m^2 - 3m = 0$

$9\left[x^2 - \frac{6}{9}mx + \frac{m^2 - 3m}{9}\right] = 0$

$\therefore x^2 - \frac{2}{3}mx + \frac{1}{9}m^2 = \frac{3m - m^2}{9} + \frac{m^2}{9}$

$\therefore \left(x - \frac{m}{3}\right)^2 = \frac{3m}{9}$

$\therefore x - \frac{m}{3} = \pm\sqrt{\frac{m}{3}}$

$\therefore x = \frac{m}{3} \pm \sqrt{\frac{m}{3}}$

$\therefore x = \frac{m}{3} \pm \sqrt{\frac{m}{3}}$: $m \geq 0$

Lesson 7

1. $9x(x - 1) = -2 \Rightarrow 9x^2 - 9x + 2 = 0$

$a = 9, b = -9, c = 2$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(9)(2)}}{2(9)}$

$= \frac{9 \pm \sqrt{81 - 72}}{18}$

$= \frac{9 \pm \sqrt{9}}{18} = \frac{9 \pm 3}{18}$

$\therefore x = \frac{9 + 3}{18} = \frac{12}{18} = \frac{2}{3}$

or

$x = \frac{9 - 3}{18} = \frac{6}{18} = \frac{1}{3}$

2. $a = 2 \quad b = 4 \quad c = 3$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{16 - 4(6)}}{4}$

$= \frac{-4 \pm \sqrt{-8}}{4}$

3. $a = 2 \quad b = 1 \quad c = -2$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1 - 4(-4)}}{4}$

$= \frac{-1 \pm \sqrt{17}}{4}$

$\therefore x = \frac{-1 + \sqrt{17}}{4}$

or

$x = \frac{-1 - \sqrt{17}}{4}$

4. $\therefore 3x^2 + 4x - 84 = 0$

$a = 3 \quad b = 4 \quad c = -84$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{16 - 4(3)(-84)}}{2(3)}$$
$$= \frac{-4 \pm \sqrt{16 + 1\,008}}{6}$$
$$= \frac{-4 \pm \sqrt{1\,024}}{6}$$
$$= \frac{-4 \pm 32}{6}$$
$$= \frac{-2 \pm 16}{3}$$

$x = \frac{-2 + 16}{3}$ or $x = \frac{-2 - 16}{3}$

$x = \frac{14}{3}$ or $x = -\frac{18}{3}$

$x = \frac{14}{3}$ or $x = -3$

5. $24x^2b^2 + 2bx - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2b \pm \sqrt{4b^2 - 4(24b^2)(-1)}}{2(24b^2)}$$
$$= \frac{-2b \pm \sqrt{102b^2}}{48b^2}$$
$$= \frac{-2b \pm b\sqrt{102}}{48b^2}$$
$$= \frac{-2 \pm \sqrt{102}}{48b}$$

6. $36x^4 - 25x^2 + 4 = 0$

$a = 36 \quad b = -25 \quad c = 4$

$$x^2 = \frac{25 \pm \sqrt{625 - 4(36)(4)}}{72}$$
$$= \frac{25 \pm \sqrt{49}}{72}$$
$$= \frac{25 \pm 7}{72}$$

$\therefore x^2 = \frac{32}{72}$ or $x^2 = \frac{18}{72}$

$\therefore x = \pm\sqrt{\frac{16 \times 2}{36 \times 2}}$ $\quad x^2 = 4$

$x = \pm\frac{4}{9}$ or $x = \pm 2$

7. $x^2(x + m) - 2x(x + m) + (x + m) = 0$

$(x + m)(x^2 - 2x + 1) = 0$

$\therefore x = -m$ or $(x - 1)^2 = 0$

$x = 1$

8. $x^2 - 6x + 2{,}6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{6 \pm \sqrt{36 - 4(2{,}6)}}{2}$$
$$= \frac{6 \pm \sqrt{25{,}6}}{2}$$

$\therefore x = 5{,}53$ or $x = 0{,}47$

9. $(x + 3)(2 - x) = 7x$

$\therefore -(x + 3)(x - 2) = +7x$

$-x^2 - x + 6 - 7x = 0$

$\therefore x^2 + 8x - 6 = 0$

$$x = \frac{-8 \pm \sqrt{64 - 4(-6)}}{2}$$
$$= \frac{-8 \pm \sqrt{88}}{2}$$

$\therefore x = 0{,}69$ or $x = -8{,}69$

Lesson 8

Activity 1

1. Let $k = 3x^2 - x : k \neq 0$

$\therefore k + \frac{6}{k+2} = 5$

$\therefore k^2 + 2k + 6 = 5k + 10$

$\therefore k^2 - 3k - 4 = 0$

$(k + 1)(k - 4) = 0$

$\therefore 3x^2 - x + 1 = 0$ or $3x^2 - x - 4 = 0$

$\therefore x = \frac{1 \pm \sqrt{1 - 3}}{6}$ $\quad$ $(3x - 4)(x + 1)$

$x = \frac{1 \pm \sqrt{-2}}{6}$ $\quad$ $x = \frac{4}{3}$ or $x = -1$

$\therefore x$ is non $\mathbb{R}$.

2. Let $k = x^2 - 5x$;

$k^2 - 2k - 24 = 0$

$(k + 4)(k - 6) = 0$

$x^2 - 5x + 4 = 0$ or $x^2 - 5x - 6 = 0$

$(x - 1)(x - 4) = 0$ $\quad$ $(x - 6)(x + 1) = 0$

$x = 1$ or $x = 4 \therefore x = 6$ or $x = -1$

3. $\sqrt{3x - 1} + 1 = \frac{6}{\sqrt{3x - 1}}$

If $k = \sqrt{3x - 1}$; $k > 0$

$k + 1 = \frac{6}{k}$

$\therefore k^2 + k - 6 = 0$

$(k + 3)(k - 2) = 0$

$\therefore \sqrt{3x - 1} + 3 = 0$ or $\sqrt{3x - 1} - 2 = 0$

$\therefore \sqrt{3x - 1} = -3$ $\quad$ $3x - 1 = 4$

impossible $\quad$ $3x = 5$

$x = \frac{5}{3}$

4. Let $k = (x^2 - 3x)$

$k^2 - 5k + 4 = 0$

$(k - 4)(k - 1) = 0$

so $x^2 - 3x - 4 = 0$ or $x^2 - 3x - 1 = 0$

$(x - 4)(x + 1) = 0$ $\quad$ $x = \frac{3 \pm \sqrt{9 - 4(-1)}}{2}$

$x = 4$ or $x = -1$ $\quad$ $= \frac{3 \pm \sqrt{13}}{2}$

$\therefore x = 3{,}30$ or $x = -0{,}30$

5. Let $k = 3x^2 + x : k \neq 0$

So $4(k + 1) - \frac{10(k - 1)}{k} = 7$

$\therefore 4k(k + 1) - 10(k - 1) = 7k$

$\therefore 4k^2 + 4k - 10k + 10 - 7k = 0$

$\therefore 4k^2 - 13k + 10 = 0$

$(4k - 5)(k - 2) = 0$

$\therefore 4(3x^2 + x) - 5 = 0$ $\quad$ $3x^2 + x - 2 = 0$

$\therefore 12x^2 + 4x - 5 = 0$ $\quad$ $(3x - 2)(x + 1) = 0$

$\therefore (2x - 1)(6x + 5) = 0$ $\quad$ $x = \frac{2}{3}$ or $x = -1$

$x = \frac{1}{2}$ or $x = -\frac{5}{6}$

6. $2x - 3 - \frac{3}{2x - 1} = 0$

Let $k = 2x - 1 : k \neq 0$

$\therefore k - 2 - \frac{3}{k} = 0$

$\therefore k^2 - 2k - 3 = 0$

$(k + 1)(k - 3) = 0$

$2x - 1 + 1 = 0$ $\quad$ $2x - 1 - 3 = 0$

$2x = 0$ $\quad$ $2x = 4$

$x = 0$ $\quad$ $x = 2$

7. $x^2 + 6x - 2 = \frac{35}{x^2 + 6x}$

Let $k = x^2 + 6x \neq 0$

$\therefore\ k - 2 = \frac{35}{k}$

$\therefore\ k^2 - 2k - 35 = 0$

$(k + 5)(k - 7) = 0$

$x^2 + 6x + 5 = 0$ or $x^2 + 6x - 7 = 0$

$(x + 1)(x + 5) = 0$ $(x - 1)(x + 7) = 0$

$x = -1$ or $x = -5$ or $x = 1$ or $x = -7$

8. $(2x^2 - 3x)^2 - 2(2x^2 - 3x) - 3 = 0$

Let $2x^2 - 3x = k$

$k^2 - 2k - 3 = 0$

$(k - 3)(k + 1) = 0$

$2x^2 - 3x = 3$ or $2x^2 - 3x = -1$

$2x^2 - 3x - 3 = 0$ or $2x^2 - 3x + 1 = 0$

$x = \frac{3 \pm\sqrt{9 - 4(2)(-3)}}{4}$ or $(2x - 1)(x - 1) = 0$

$= \frac{3 \pm\sqrt{33}}{4}$ or $x = \frac{1}{2}$ or $x = 1$

9. let $\sqrt{2x - 1} = k;\ k > 0$

$\frac{k}{2} - \frac{4}{k} - 1 = 0$

lcd $2k$

$k2 - 8 - 2k = 0$

$k2 - 2k - 8 = 0$

$(k - 4)(k + 2) = 0$

$k = 4$ or $k = -2$

$\sqrt{2x - 1} = 4$ or n.a

$2x - 1 = 16$

$2x = 17$

$x = \frac{17}{2}$

10. $x^2 - \frac{5}{x^2 + x} = 4 - x$

$\therefore\ x^2 + x - \frac{5}{x^2 + x} = 4$

Let $k = x^2 + x \neq 0$

$\therefore\ k - \frac{5}{k} = 4$

$\therefore\ k^2 - 4k - 5 = 0$

$(k - 5)(k + 1) = 0$

$x^2 + x - 5 = 0$

$x = \frac{-1 \pm \sqrt{1 + 20}}{2} = \frac{-1 \pm \sqrt{21}}{2}$

$\therefore\ x = 1{,}79$ or $x = -2{,}79$

or $x^2 + x + 1 = 0$

$x = \frac{-1 \pm \sqrt{1 - 4}}{2a}$ non $\mathbb{R}$

Activity 2

Solution no 1:

⇒ Let the digits be x and y.

x – the tens digit

y – the units digit

$\therefore\ 7(x + y) = 10x + y$ … (1)

$xy - x = \frac{12x}{y}$ … (2)

From (1): $7x + 7y = 10x + y$

$\therefore\ -3x = -6y$

$x = 2y$ … (3)

(3) → (2): $2y.y - 2y = \frac{12.2y}{y}$

$\therefore\ 2y^2 - 2y - 24 = 0$

$\therefore\ y^2 - y - 12 = 0$

$\therefore\ (y - 4)(y + 3) = 0$

$\therefore\ y = 4$ or $y = -3$

If $y = 4$, then $x = 2(4)$ or if $y = -3$, then $x = 2(-3)$

$x = 8$ $x = -6$

Solution to no 2

⇒ Let the one integer be x and the other y.

Then the consecutive integer (y) will be $y = x +$
1

Thus: $xy = 90$

$x(x + 1) = 90$

$\therefore\ x^2 + x - 90 = 0$

$\therefore\ (x + 10)(x - 9) = 0$

$\therefore\ x = -10$ or $x = 9$

Thus the integers are –10 and –9 or 9 and 10.

1. $3^{2x} - 12 \cdot 3^x + 27 = 0$

$(3^x = k) \Rightarrow \quad k^2 - 12k + 27 = 0$

$(k - 9)(k - 3) = 0$

$3^x = 9 \quad 3^x = 3$

$x = 2 \qquad x = 1$

2. $3^x \cdot 3 - 10 \cdot 3^x \cdot 3^{-1} + 3 = 0$

$(3^x = k) \Rightarrow 3k - \frac{10}{3}k + 3 = 0$

$\Rightarrow 9k - 10k + 9 = 0$

$k = 9$

$x = 2$

3. Let $2^{\frac{x}{2}} = k$

$\left(k - \frac{4}{k}\right)\left(k - \frac{8}{k}\right) = 0$

$\Rightarrow (k^2 - 4)(k^2 - 8) = 0$

$k^2 = 4 \quad k^2 = 8$

$\left(2^{\frac{x}{2}}\right)^2 = 2^2 \qquad \left(2^{\frac{x}{2}}\right)^2 = 2^3$

$x = 2 \quad x = 3$

4. Let $2^{\frac{2x}{3}} = k$

$\Rightarrow 4k - 33 + \frac{8}{k} = 0$

$\Rightarrow 4k^2 - 33k + 8 = 0$

$\Rightarrow (4k - 1)(k - 8) = 0$

$k = \frac{1}{4} \qquad k = 8$

$2^{\frac{2x}{3}} = 2^{-2} \qquad 2^{\frac{2x}{3}} = 2^3$

$\Rightarrow \frac{2x}{3} = -2 \qquad \frac{2x}{3} = 3$

$\Rightarrow x = -3 \qquad x = 4\frac{1}{2}$

5. $4 \cdot 2^{2x} \cdot 4 - 17 \cdot 2^x + 1 = 0$

Let $2^x = k$

$16k^2 - 17k + 1 = 0$

$(16k - 1)(k - 1) = 0$

$k = \frac{1}{16} \; k = 1$

$2^x = 2(-4) \qquad 2^x = 2^0$

$x = -4 \; x = 0$

6. Let $2^x = k$

$8k^2 + 10k - 3 = 0$

$(4k - 1)(2k + 3) = 0$

$k = \frac{1}{4} \qquad k = -\frac{3}{2}$

$2^x = 2^{-2} \qquad$ invalid

$x = -2$

Lesson 9

Activity 1

1. $y = 9 - x \rightarrow x^2 + x(9 - x) + (9 - x)^2 = 61$

$\therefore \; x^2 + 9x - x^2 + 81 - 18x + x^2 - 61 = 0$

$\therefore \; x^2 - 9x + 20 = 0$

$(x - 4)(x - 5) = 0$

$x = 4$ or $x = 5$

$y = 9 - 4 = 5 \quad y = 9 - 5 = 4$

$\therefore$ $(4; 5)$ or $(5; 4)$

2. $x - 2y = 3$

$x = 2y + 3 \rightarrow 4(2y + 3)^2 - 5y(2y + 3) = 3 - 6y$

$\therefore \; 4(4y^2 + 12y + 9) - 10y^2 - 15y = 3 - 6y$

$\therefore \; 16y^2 + 48y + 36 - 10y^2 - 9y - 3 = 0$

$\therefore \; 6y^2 + 39y + 33 = 0$

$\therefore \; 2y^2 + 13y + 11 = 0$

$(2y + 11)(y + 1) = 0$

$\therefore \; y = -\frac{1}{2}$ or $y = -1$

$x = 2\left(-\frac{11}{2}\right) + 3$ or $x = 2(-1) + 3$

$= -11 + 3 \qquad x = 1$

$x = -8$

$\left(-8; -\frac{1}{2}\right)$ or $(1; -1)$

3. $(x-1)^2 + (y-2)^2 = 5 \qquad y = -2x - 1$

$\therefore\ (x-1)^2 + (-2x-1-2)^2 = 5$

$\therefore\ x^2 - 2x + 1 + 4x^2 + 12x + 9 - 5 = 0$

$\therefore\ 5x^2 + 10x + 5 = 0$

$\therefore\ x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$\therefore\ x = -1$

$y = 2 - 1 = 1$

$(-1; 1)$

4. $(x-2)^2 + (y-3)^2 = 0$

Since $a^2 + b^2 > 0 \quad a; b \in \mathbb{R}$, this equation will equal zero if and only if $x = 2$ and $y = 3$

5. $(x^2 + 2)(y - 3) = 0$

$a.b = 0$ if and only if one of a or b is equal to zero.

So $x^2 + 2 = 0$ or $\qquad y - 3 = 0$

$x^2 = -2 \qquad\qquad y = 3$ only

rom $\mathbb{R}$

6. $2x - 3y = 1$

$2x = 3y + 1$

$x = \frac{3y+1}{2} \rightarrow x(x-6) - y = (1-y)(1+y)$

$\therefore\ \left(\frac{3y+1}{2}\right)\left(\frac{3y+1}{2} - 6\right) - y = 1 - y^2$

$\left(\frac{3y+1}{2}\right)^2 - 6\left(\frac{3y+1}{2}\right) - y = 1 - y^2$

$\therefore\ \frac{9y^2 + 6y + 1}{4} - 3(3y+1) - y = 1 - y^2$

$\therefore\ 9y^2 + 6y + 1 - 12(3y+1) - 4y = 4 - 4y^2$

$\therefore\ 9y^2 + 6y + 1 - 36y - 12 - 4y - 4 + 4y^2 = 0$

$\therefore\ 13y^2 - 34y - 15 = 0$

$\therefore\ (13y + 5)(y - 3) = 0$

$y = -\frac{5}{13}$ or $y = 3$

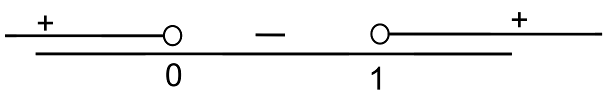
so $x = \frac{3\left(\frac{-5}{13}\right) + 1}{2}$ or $x = \frac{3(3)+1}{2}$

$x = -\frac{1}{13} \qquad x = 5$

Lesson 10

Activity 1

1. $x(x-1) > 0$



$x < 0 \quad$ or $\quad x > 1$

2. $x \in (-2; 3]$

3. $x \le -2 \quad$ or $\quad x \ge 4$

4. $x < 0 \quad$ or $\quad x > 4$

5. $0 \le x \le 3$

Activity 2

1. $[-3; 0] \quad$ or $\quad [4; \infty)$

2. $(-\infty; -4] \quad$ or $\quad (5; \infty)$

3. $[-2; 0) \quad$ or $\quad (1; 3]$

4. $(-6; -1) \quad$ or $\quad [0; 2]$

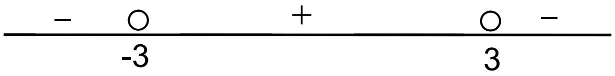
5. $(-\infty ; -1)$ or $(1 ; 4]$ 6. $(-1 ; 0)$ or $(2 ; \infty)$

Activity 3

1. $9 - k^2 \geq 0$

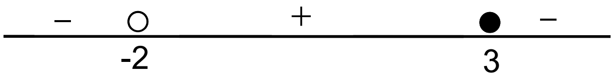
   $(3 - k)(3 + k) \geq 0$

   $k \in [-3 ; 3]$



2. $\frac{3 - p}{2 + p} \geq 0$

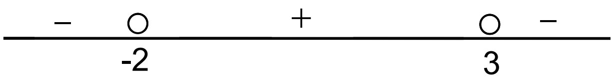
   $p \in (-2 ; 3]$



3. $6 + m - m^2 < 0$ or $m - 1 = 0$

   $(3 - m)(2 + m) < 0$

   $m < -2$ or $m > 3$ or $m = 1$



4. $\frac{-(t-1)^2}{2t} \geq 0$ $t = 1$

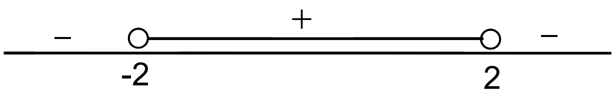
   $2t < 0$

   $t < 0$ or $t = 1$

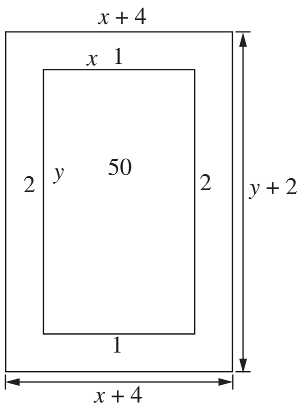
5. $4 - p^2 \geq 0$ but $p \neq 1$

   $(2 - p)(2 + p) \geq 0$

   $p \in [-2 : 2]$ but $p \neq 1$



Lesson 11

1. 

$xy = 50$ … (1) (area)

$2(x + 4) + 2(y + 2) = 42$ (perimeter)

$x + 4 + y + 2 = 21$

$x + y = 15$

$\therefore\ y = 15 - x$ … (2)

$\therefore\ x(15 - x) = 50$

$-x^2 + 15x - 50 = 0$

$\therefore\ x^2 - 15x + 50 = 0$

$(x - 5)(x - 10) = 0$

$x = 5$ or $x = 10$

Then: $y = 15 - 5 = 10$ or $y = 15 - 10 = 5$

$\therefore$ Dimensions of card: L = 5 + 4 = 9

B = 10 + 2 = 12

or L = 10 + 4 = 14

B = 5 + 2 = 7

So (9 × 12) or (7 × 14)

2. Let the numbers be x and y.

$x - y = 3$ $(x > y)$

$xy = 40$ (Product)

So $x = y + 3 \rightarrow (y + 3)y = 40$

$\therefore\ y^2 + 3y - 40 = 0$

$\therefore\ (y - 5)(y + 8) = 0$

$\therefore\ y = 5$ or $y = -8$

$x = 8$ n.a.

The two numbers are 8 and 5.

3. Let the speed of the old locomotive be x km/h.

	S	t	D
Old	x	$\frac{100}{x}$	100
New	$x + 10$	$\frac{100}{x + 10}$	100

$T_{slow} = T_{fast} + \frac{1}{2}$ hour

$\therefore\ \frac{100}{x} = \frac{100}{x + 10} + \frac{1}{2}$

$\therefore\ 200(x + 10) = 200x + x(x + 10)$

$\therefore\ 200x + 2\,000 = 200x + x^2 + 10x$

$\therefore\ x^2 + 10x - 2\,000 = 0$

$(x + 50)(x - 40) = 0$

$x = 40$ km/h

4. Let the apprentice take x days.

Then the electrician takes $(x - 9)$ days.

$\frac{1}{x} + \frac{1}{x-9} = \frac{1}{20}$ Restriction: $x - 9 > 0$

$20(x - 9) + 20x = x^2 - 9x$ $\therefore x > 9$

$\therefore 20x - 180 + 20x = x^2 - 9x$

$\therefore x^2 - 49x + 180 = 0$

$\therefore (x - 45)(x - 4) = 0$

$\therefore x = 45$ or $x = 4$

n.a.

The apprentice takes 45 days and the electrician takes 36 days.

5. Let the stream flow be x km/h (x > 0)

	S	D	t
Up	$30 - x$	12	$\frac{12}{30-x}$
Down	$30 + x$	12	$\frac{12}{30+x}$

$T_{up} + T_{down} = 1$ hour

$\therefore \frac{12}{30-x} + \frac{12}{30+x} = 1$

$\therefore 12(30 + x) + 12(30 - x) = 900 - x^2$

$\therefore 360 + 12x + 360 - 12x = 900 - x^2$

$\therefore x^2 = 180$

$\therefore x = \pm\sqrt{180}$

$\therefore x = \sqrt{36 \times 5} = 6\sqrt{5}$ $(x > 0)$

$\therefore$ The stream flowed at $6\sqrt{5}$ km/h.

6.

	D (km)	S (km/h)	t (min)
Clear	$1{,}5 + x$	S_1	$\frac{5x}{2}$
Rainy	$1{,}5 + x$	52	$\frac{15x}{4}$

20 km/h $= \frac{20}{60}$ km/min

$S_1 = S_2 + \frac{1}{3}$

$\frac{1{,}5 + x}{\frac{5x}{2}} = \frac{1{,}5 + x}{\frac{15x}{4}} + \frac{1}{3}$

$\therefore \frac{3 + 2x}{5x} = \frac{6 + 4x}{15x} + \frac{1}{3}$

$\therefore 3(3 + 2x) = 6 + 4x + 5x$

$\therefore 9 + 6x = 6 + 9x$

$\therefore 3x = 3$

$x = 1$ km

The train is 1 km long.

7. Let the denominator be x and numerator be y:

$x = 1 + y^2$ … (1) $x > 0$ or $x < 0$

$y > 0$ $y < 0$

$\frac{y+1}{x-3} = \frac{1}{4}$ … (2)

$\therefore 4y + 4 = x + 3$

$= 1 + y^2 + 3$

$\therefore y^2 + 4 = 4y + 4$

$\therefore y^2 - 4y = 0$

$y(y - 4) = 0$

$\therefore y = 0$ or $y = 4$

n.a.

Then $x = 1 + 16 = 17$

So the fraction is $\frac{4}{17}$.

8. a) Each person pays R$\frac{120}{x}$

b) R$\left(\frac{120}{x-4}\right)$ each

c) $\frac{120}{x} + 5 = \frac{120}{x-4}$

$\therefore\ 120(x-4) + 5x(x-4) = 120x$

$\therefore\ 120x - 480 + 5x^2 - 20x - 120x = 0$

$\therefore\ 5x^2 - 20x - 480 = 0$

$\therefore\ x^2 - 4x - 96 = 0$

$\therefore\ (x-12)(x+8) = 0$

$\therefore\ x = 12$ or $x = -8$

n.a.

There were 12 people in the original group.

9. Assume he bought x sheep: $(x > 0)$

He paid $\frac{960}{x}$ rand per sheep.

He had $(x-4)$ to sell.

So: $\left(\frac{960}{x} + 8\right)$ is the selling price per sheep.

$\therefore\ \left(\frac{960}{x-4}\right) = \frac{960}{x} + 8$

So $960x = 960(x-4) + 8x(x-4)$

$\therefore\ 960x = 960x - 3\,840 + 8x^2 - 32x$

$\therefore\ 8x^2 - 32x - 3\,840 = 0$

$\therefore\ x^2 - 4x - 480 = 0$

$\therefore\ (x+20)(x-24) = 0$

$\therefore\ x = 24$

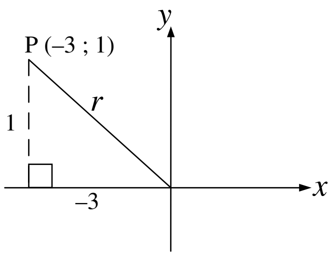
$\therefore$ He bought 24 sheep.

Lessons 12–13

1. (a) 2nd (b) 2nd (c) 3rd

(d) 3rd (e) 1st (f) 3rd

2.



$x = -3 \quad y = 1 \quad r^2 = 10$

$r = \sqrt{10}$

(a) $\frac{1}{\sqrt{10}} - \frac{3}{\sqrt{10}} = -\frac{2}{\sqrt{10}}$

(b) $1 + \left(\frac{1}{\sqrt{10}}\right)^2 = 1 + \frac{1}{10} = \frac{11}{10}$

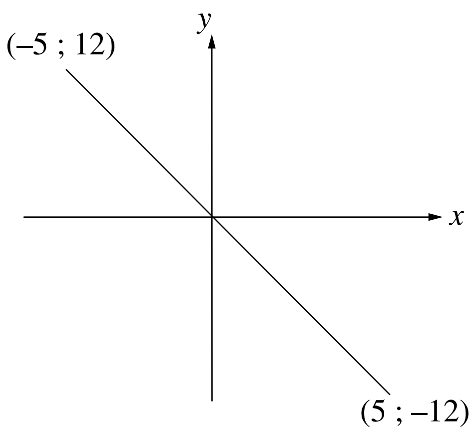
(c) $\sin^2\theta + \cos^2\theta$

$= \left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{-3}{\sqrt{10}}\right)^2 = 1$

3. $\tan\theta = -\frac{5}{12}$

$r^2 = 25 + 144$

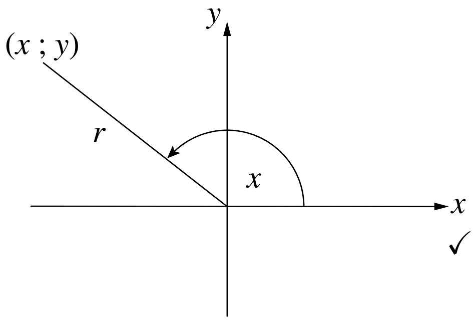
$r = 13$



a) $\sin\theta = \pm\,\frac{12}{13}$

b) $\cos\theta = \pm\,\frac{5}{13}$

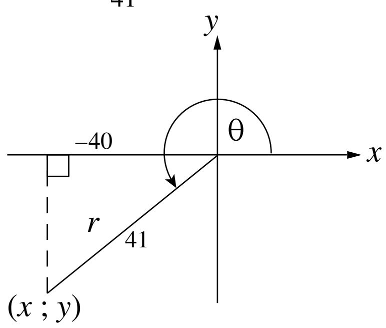
4. $\cos x = \frac{-24}{25}$



$\dfrac{2\left(\frac{-7}{24}\right)}{\frac{7}{25}} = \frac{-14}{24} \times \frac{25}{7} = \frac{-25}{12}$

$x = -24 \quad y = +7 \quad r = +25$

5. $\cos \theta = \frac{-40}{41}$

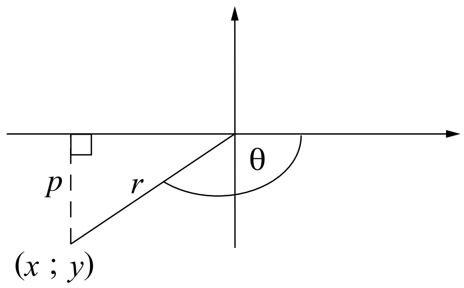


(a) $1 + \left(-\frac{9}{40}\right)^2 = 1 + \frac{81}{1600}$
$= \frac{1681}{1600}$

(b) $2\left(\frac{18}{41}\right) - \left(\frac{-40}{41}\right) = \frac{76}{41}$

$x = -40 \quad y = +9 \quad r = +41$

6. $\sin \theta = \frac{\text{p}}{1}$



$\cos \theta = -\sqrt{1 - \text{p}^2}$

$y = -\text{p} \quad r = 1 \quad x = -\sqrt{1 - \text{p}^2}$

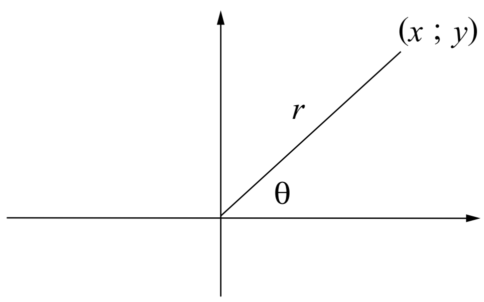
7. $\tan \theta = \frac{m}{1}$

$\sin^2 \theta - \cos^2 \theta$

$= \left(\frac{m}{\sqrt{m^2 + 1}}\right)^2 - \left(\frac{1}{\sqrt{m^2 + 1}}\right)^2$

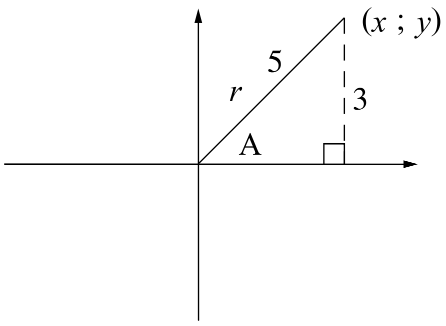
$= \frac{m^2}{m^2 + 1} - \frac{1}{m^2 + 1}$

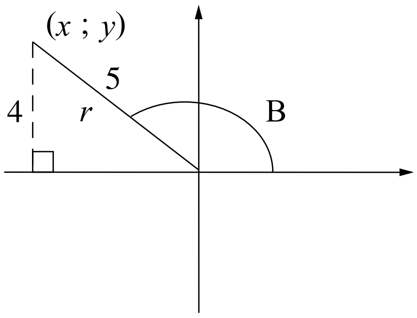
$= \frac{m^2 - 1}{m^2 + 1}$



$y = m \quad x = 1 \quad r^2 = m^2 + 1$
$r = \sqrt{m^2 + 1}$

8.





$x = +4$
$y = +3$
$r = +5$

$x = -3$
$y = +4$
$r = +5$

$$\frac{\left(\frac{4}{5}\right) + \left(\frac{-3}{5}\right)}{\left(\frac{3}{4}\right) - \left(\frac{-4}{3}\right)} = \frac{48 - 36}{45 + 80} = \frac{12}{125}$$

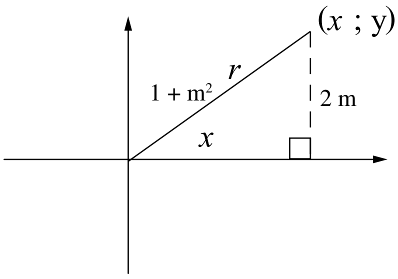
9. $\sin x = \dfrac{2m}{1 + m^2}$

$\dfrac{1 + m^2}{1 - m^2} + \dfrac{2m}{1 - m^2}$

$= \dfrac{1 + 2m + m^2}{(1 - m^2)}$

$= \dfrac{(1 + m)^2}{(1 - m)(1 + m)}$

$= \dfrac{1 + m}{1 - m}$



$y = 2\text{m} \qquad r = (1 + \text{m}^2)$

$x^2 = (1 + \text{m}^2)^2 - 4\text{m}^2$

$x^2 = 1 + 2\text{m}^2 + \text{m}^4 - 4\text{m}^2$

$x^2 = 1 - 2\text{m}^2 + \text{m}^4$

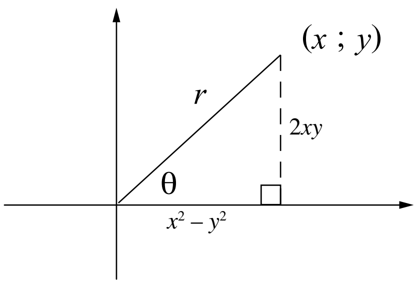
$x^2 = (1 - 1\text{m}^2)^2$

$x = 1 - 1\text{m}^2$

10. $\tan \theta = \dfrac{2xy}{x^2 - y^2}$

$\dfrac{2xy}{x^2 + y^2} + \dfrac{x^2 - y^2}{x^2 + y^2}$

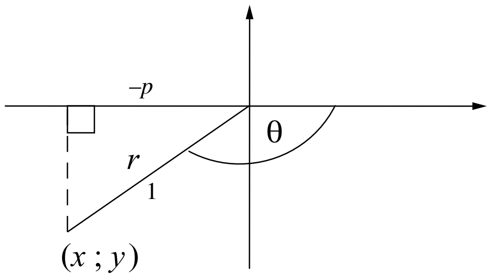
$= \dfrac{2xy + x^2 - y^2}{x^2 + y^2}$



$y = 2xy \quad x = x^2 - y^2 \quad r = x^2 + y^2$

11. $\cos\theta = \frac{p}{1}$

$\tan\theta = \frac{-\sqrt{1-p^2}}{p}$



$x = pr = 1y = -\sqrt{1-p^2}$

Lesson 14

1. $\frac{\sin^2\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta} = \frac{\sin^2\theta - 1}{\cos^2\theta} = \frac{-\cos^2\theta}{\cos^2\theta} = -1$

2. $(1 - \sin^2 x) + \cos^2 x = \cos^2 x + \cos^2 x = 2\cos^2 x$

3. $\frac{1}{\sin A} - \cos A \times \frac{\cos A}{\sin A} = \frac{1-\cos^2 A}{\sin A} = \frac{\sin^2 A}{\sin A} = \sin A$

4. $\frac{\sin^2\alpha}{\cos^2\alpha} \times \cos^2\alpha + \sin^2\alpha \times \frac{\cos^2\alpha}{\sin^2\alpha} = 1$

5. LHS

$\frac{\sin A \times \cos^2 A}{\cos A \sin^2 A}$

$= \frac{\cos A}{\sin A}$

RHS

$= \frac{\cos A}{\sin A}$

6. LHS $\cos^2\theta\left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right) = \frac{\cos^2\theta}{1}\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)$

$= 1$

7. $(\sin\ \alpha + \cos\alpha)^2$

$= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha$

$= 1 + 2\sin\alpha\cos\alpha = \text{RHS}$

8. LHS

$(2\cos^2\theta - 1)\left(\frac{1+\sin^2\theta}{\cos^2\theta}\right)$

$= (2\cos^2\theta - 1)\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)$

$= (2\cos^2\theta - 1)\left(\frac{1}{\cos^2\theta}\right)$

$= 2 - \frac{1}{\cos^2\theta} - \frac{2\cos^2\theta - 1}{\cos^2\theta}$

LHS = RHS

RHS

$\frac{1-\sin^2\theta}{\cos^2\theta = \cos^2\theta - \sin^2\theta/\cos^2\theta}$

$= \frac{\cos^2\theta - (1-\cos^2\theta)}{\cos^2\theta}$

$= \frac{2\cos^2\theta - 1}{\cos^2\theta}$

9. LHS $\frac{1-\sin\alpha + 1 + \sin\alpha}{(1+\sin\alpha)(1-\sin\alpha)} = \frac{2}{1-\sin^2\alpha} = \frac{2}{\cos^2\alpha} = \text{RHS}$

10. LHS $\sqrt{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta} = \sqrt{(\sin\theta + \cos\theta)^2} = \sin\theta + \cos\theta$

11. LHS $\dfrac{\frac{1-\sin x}{\cos x}}{\frac{1+\sin x}{\cos x}}$ ($\times$ all terms by $\cos x$)

$= \frac{\cos x - \sin x}{\cos x + \sin x} = \text{RHS}$

12. LHS $\dfrac{\frac{\sin^2\alpha}{\cos^2\alpha}}{1 + \frac{\sin^2\alpha}{\cos^2\alpha}}$ ($\times \cos^2\alpha$)

$= \frac{\sin^2\alpha}{\cos^2\alpha + \sin^2\alpha} = \sin^2\alpha = \text{RHS}$

13. LHS

$\dfrac{\sin^2 A + (1+\cos A)^2}{(1+\cos A)\sin A}$

$= \dfrac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{(1+\cos A)\sin A}$

$= \dfrac{2 + 2\cos A}{(1+\cos A)\sin A}$

$= \dfrac{2(1+\cos A)}{(1+\cos A)\sin A} = \dfrac{2}{\sin A} = \text{RHS}$

14. LHS

$\left(\dfrac{1-\sin^2 x}{\sin x}\right)^2$

$= \left(\dfrac{\cos^2 x}{\sin x}\right)^2$

$= \dfrac{\cos^4 x}{\sin^2 x}$

RHS

$\dfrac{\cos^2 x}{\sin^2 x} - \dfrac{\cos^2 x}{1}$

$= \dfrac{\cos^2 x - \cos^2 x\sin^2 x}{\sin^2 x}$

$= \dfrac{\cos^2 x(1-\sin^2 x)}{\sin^2 x}$

$= \dfrac{\cos^2 x\cos^2 x}{\sin^2 x}$

$= \dfrac{\cos^4 x}{\sin^2 x}$

15. RHS $\dfrac{\sin\theta}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}$

$= \dfrac{1-\cos^2\theta - \cos^2\theta}{\cos\theta\sin\theta}$

$= \dfrac{1-2\cos^2\theta}{\cos\theta\sin\theta}$

= LHS (Sometimes choose the RHS)

16. LHS $\dfrac{1}{\sin^2 A} - \dfrac{\cos A}{\sin^2 A}$

$= \dfrac{1-\cos A}{\sin^2 A}$

$= \dfrac{1-\cos A}{1-\cos^2 A}$

$= \dfrac{(1-\cos A)}{(1-\cos A)(1+\cos A)} = \dfrac{1}{(1+\cos A)} = \text{RHS}$

17. LHS

$\dfrac{\sin^2\alpha}{\cos^2\alpha} - \dfrac{\sin^2\alpha}{1}$

$= \dfrac{\sin^2\alpha - \sin^2\alpha\cos^2\alpha}{\cos^2\alpha}$

$= \dfrac{\sin^2\alpha(1-\cos^2\alpha)}{\cos^2\alpha}$

$= \dfrac{\sin^2\alpha\sin^2\alpha}{\cos^2\alpha}$

$= \dfrac{\sin^4\alpha}{\cos^2\alpha}$

RHS

$\dfrac{\sin^2\alpha}{\cos^2\alpha} \times \sin^2\alpha$

$= \dfrac{\sin^4\alpha}{\cos^2\alpha}$

18. LHS $\dfrac{1-\cos^2\alpha}{1+\cos\alpha} = \dfrac{(1-\cos\alpha)(1+\cos\alpha)}{1+\cos\alpha}$

$= 1 - \cos\alpha = \text{RHS}$

19. LHS $\dfrac{\sin x}{1-\cos x}$ (× top and bottom by $\sin x$)

$= \dfrac{\sin^2 x}{(1-\cos x)\sin x}$

$= \dfrac{1-\cos^2 x}{(1-\cos x)\sin x} = \dfrac{(1-\cos x)(1+\cos x)}{(1-\cos x)\sin x}$

$= \dfrac{1+\cos x}{\sin x}$

20. LHS $\left(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right)^2 = \left(\dfrac{1-\sin x}{\cos x}\right)^2$

$= \dfrac{(1-\sin x)^2}{\cos^2 x}$

$= \dfrac{(1-\sin x)^2}{1-\sin^2 x}$

$= \dfrac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$

$= \dfrac{1-\sin x}{1+\sin x} =$ RHS

Lesson 15

Activity 1

A.

1) $1 - \frac{\sin^2 A}{1 - (-\cos A)}$

$= 1 - \frac{1 - \cos^2 A}{1 + \cos A}$

$= \frac{1 + (1 + \cos A)(1 - \cos A)}{(1 + \cos A)}$

$= \cos A$

2) $\frac{(\sin\theta)(\sin\theta)(-\cos\theta)}{(\cos\theta)\left(\frac{\sin\theta}{\cos\theta}\right)(\sin\theta)} = \cos\theta$

3) $= \frac{1}{-\left(\frac{\sin\theta}{\cos\theta}\right)(-\cos\theta)} - \frac{\cos^2\theta}{\sin\theta}$

$= \frac{1}{\sin\theta} - \frac{\cos^2\theta}{\sin\theta}$

$= \frac{1 - \cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta}$

$= \sin\theta$

4) $\frac{\sin^2\theta + \cos^2\theta}{-\left(\frac{\sin\theta}{\cos\theta}\right)/\cos\theta}$

$= \frac{1}{\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{1}\right)}$

$= + \frac{1}{\sin\theta}$

5) $\frac{\sin A}{\cos A} \cdot \frac{1}{\sin A} - \frac{\sin^2 A}{\cos A}$

$= \frac{1 - \sin^2 A}{\cos A}$

$= \frac{\cos^2 A}{\cos A} = \cos A$

6) $\frac{\tan\theta - \sin\theta}{1 - \cos\theta}$

$= \frac{\frac{\sin\theta}{\cos\theta} - \sin\theta}{1 - \cos\theta}$

$= \frac{\sin\theta - \sin\theta\cos\theta}{\cos\theta(1 - \cos\theta)}$

$= \frac{\sin\theta(1 - \cos\theta)}{\cos\theta(1 - \cos\theta)}$

$= \tan\theta$

7) $\frac{(\sin\theta)\sin\theta - \cos^2\theta}{(\tan\theta) + \frac{1}{\tan\theta}}$

$= \frac{\sin^2\theta - \cos^2\theta}{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}$

$= \frac{(\sin^2\theta - \cos^2\theta)}{1}$

$\times \frac{\sin\theta\cos\theta}{(\sin^2\theta - \cos^2\theta)}$

$= \sin\theta\cos\theta$

B.

1) LHS

$\frac{1}{\sin A + 1} - \frac{1}{\sin A - 1}$

$= \frac{\sin A - 1 - (\sin A + 1)}{(\sin A + 1)(\sin A - 1)}$

$= \frac{-2}{\sin^2 A - 1}$

$= \frac{2}{1 - \sin^2 A}$

$= \frac{2}{\cos^2 A}$

2) LHS

$2(-\cos x)(-\sin x)$

$= 2 \sin x \cos x$

$\therefore$ LHS $=$ RHS

RHS

$$\frac{2(\tan x)}{1 + \tan^2 x} = \frac{\frac{2 \sin x}{\cos x}}{1 + \frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$$

$= 2 \sin x \cos x$

3) LHS

$$\left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta}\right)^2$$

$$= \left(\frac{1 - \sin \theta}{\cos \theta}\right)^2$$

$$= \frac{(1 - \sin \theta)^2}{\cos^2 \theta}$$

$$= \frac{(1 - \sin \theta)^2}{(1 - \sin^2 \theta)}$$

$$= \frac{(1 - \sin \theta)^2}{(1 - \sin \theta)(1 + \sin \theta)}$$

$$= \frac{1 - \sin \theta}{1 + \sin \theta}$$

$\therefore$ LHS $=$ RHS

RHS

$$\frac{1 + (-\sin \theta)}{1 - (-\sin \theta)}$$

$$= \frac{1 - \sin \theta}{1 + \sin \theta}$$

4) LHS

$$\left(\frac{1}{\cos \theta}\right) - (-\tan \theta)$$

$$= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}$$

$$= \frac{1 + \sin \theta}{\cos \theta}$$

RHS

$$\frac{\cos \theta}{1 - \sin \theta}$$

$$= \frac{\cos \theta \times \cos \theta}{(1 - \sin \theta) \times \cos \theta}$$

$$= \frac{\cos^2 \theta}{\cos \theta(1 - \sin \theta)}$$

$$= \frac{1 - \sin^2 \theta}{\cos \theta(1 - \sin \theta)}$$

$$= \frac{(1 - \sin \theta)(1 + \sin \theta)}{\cos \theta(1 - \sin \theta)}$$

$$= \frac{1 + \sin \theta}{\cos \theta}$$

Activity 2

A

1) $$\frac{(-\sin A)(\tan A)(\cos A)}{(-\cos A)(\sin A)(-\tan A)} = -1$$

2) $$\frac{(\cos \theta)(-\sin \theta)(-\sin \theta)}{(-\tan \theta)(\cos \theta)(\tan \theta)}$$

$$= -\frac{\sin^2 \theta}{1} \times \frac{\cos^2 \theta}{\sin^2 \theta}$$

$= -\cos^2 \theta$

3) $$\frac{(-\sin \alpha)(\cos \alpha)(-\sin \alpha)}{(-\cos \alpha)(-\tan \alpha)(\cos \alpha)}$$

$$= \frac{\sin^2 \alpha \cos\alpha}{\cos \alpha \sin \alpha}$$

$= \sin \alpha$

4) $\sin \theta \dfrac{\sin \theta}{\cos \theta} \cos \theta - \cos \theta(-\cos \theta)$

$= \sin^2 \theta + \cos^2 \theta$

$= 1$

5) $\dfrac{(\sin\beta)(-\tan\beta)(\cos^2\beta)}{(\cos\beta)(\sin\beta)(\cos\beta)}$

$= -\tan\beta$

6) $\dfrac{(\tan\theta)(\sin\theta)}{\cos\theta} - \dfrac{(-\cos\theta)}{\cos\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta} + 1$

$= \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}$

$= \dfrac{1}{\cos^2\theta}$

B

1) LHS $\dfrac{\sin^2 x + \cos^2 \mathrm{x}}{(\tan x)(-\cos x)}$

$= \left(\dfrac{1}{\sin x/\cos x}\right)(-\cos x)$

$= -\dfrac{1}{\sin x}$

2) LHS $\left(\dfrac{(\cos\alpha)(-\cos\alpha)}{\sin\alpha}\right)^2$

$= \dfrac{\cos^4\alpha}{\sin^2\alpha}$

RHS $\dfrac{\cos^2\alpha}{\sin^2\alpha} - \cos^2\alpha$

$= \dfrac{\cos^2\alpha - \sin^2\alpha\cos^2\alpha}{\sin^2\alpha}$

$= \dfrac{\cos^2\alpha(1 - \sin^2\alpha)}{\sin^2\alpha}$

$= \dfrac{\cos^4\alpha}{\sin^2\alpha}$

3) LHS $\dfrac{(-\sin x)(\tan x)(\cos x)}{(-\sin x)(\cos x)}$

$= \tan x$

4) LHS $\dfrac{1}{\sin \mathrm{A} + 1} - \dfrac{1}{\sin \mathrm{A} - 1}$

$= \dfrac{\sin\mathrm{A} - 1 - \sin\mathrm{A} - 1}{(\sin\mathrm{A} + 1)(\sin\mathrm{A} - 1)}$

$= \dfrac{-2}{\sin^2\mathrm{A} - 1}$

$= \dfrac{-2}{-\cos^2\mathrm{A}}$

$= \dfrac{2}{\cos^2\mathrm{A}}$

5) LHS $\cos x\left(\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}\right)$

$= \dfrac{\cos x}{1}\left(\dfrac{\sin^2 x + \cos^2 x}{\sin x\cos x}\right)$

$= \dfrac{1}{\sin x}$

Activity 3

1) a) $\sin 250°$
$= -\sin 70°$
$= -\cos 20°$

b) $\cos 340°$
$= \cos 20°$

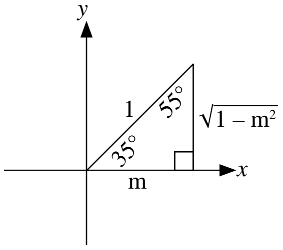
c) $\tan 160°$
$= -\tan 20°$

2) $\cos 35° = m$ $\quad \sin 55° = m$ $\quad \sin 35° = \sqrt{1 - m^2}$

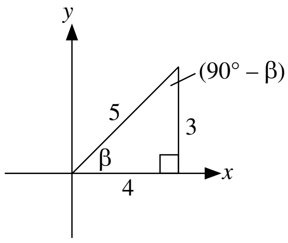
a) $\sin 305°$
$= -\sin 55°$
$= -m$

b) $\sin 245°$
$= -\sin 55°$
$= -m$

c) $\cos 245°$
$= -\cos 55°$
$= -\sqrt{1 - m^2}$



3) $x = 3 \quad y = 4 \quad r = 5$ $\qquad$ $x = 4 \quad y = 3 \quad r = 5$



$\cos \beta = \frac{4}{5}$

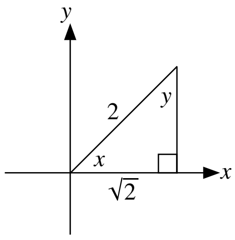
$\sin \beta - \tan \alpha$

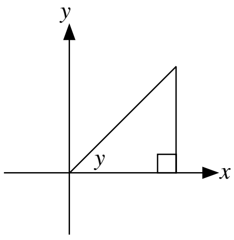
$= \frac{4}{5} - \frac{4}{3}$

$= \frac{12 - 20}{15}$

$= -\frac{8}{15}$

4) $\cos x = \frac{\sqrt{2}}{2}$ $\qquad$ $\sin y \frac{\sqrt{2}}{2}$





$x = \sqrt{2} \quad r = 2 \quad y^2 = 4 - 2$ $\qquad$ $y = \sqrt{2} \quad r = 2 \quad x = \sqrt{2}$

$y^2 = 2$

$y = \sqrt{2}$

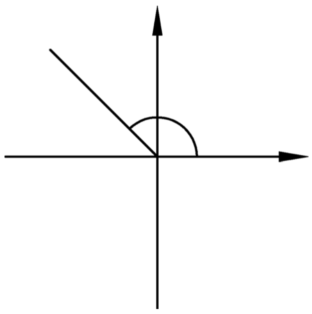
a) $\cos^2 x \circ \sin^2 y = 0$

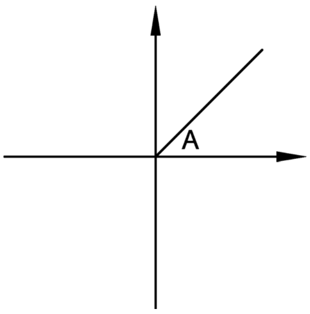
b) $\tan x \cdot \tan y$

$= 1 \times 1$

$= 1$

5) $\tan B = -\frac{8}{15}$ $\qquad$ $\tan A = \frac{8}{15}$





$x = {}^\circ 15 \quad y = +8 \quad r = 17$ $\qquad$ $x = 15 \quad y = 8 \quad r = 17$

$\sin A + \cos B = \frac{8}{17} + \frac{15}{17} = \frac{23}{17}$

6) $\cos\alpha\frac{12}{13}$ $\cos\beta\frac{12}{13}$

$x = +12$ $y = +5\ r = +13$ $x = -12$ $y = +5\ r = +13$

$\tan\beta - \sin\alpha$

$= -\frac{5}{12} - \frac{5}{13}$

$= \frac{-65-60}{156}$

$= \frac{-125}{156}$

7) $\sin 25° = p$ $\cos 25° = \sqrt{1-p^2}$ $\cos 65° = p$

a) $\cos 245°$

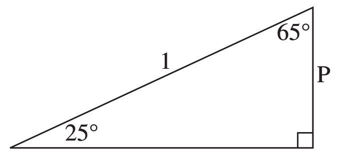
$= -\cos 65°$

$= -p$

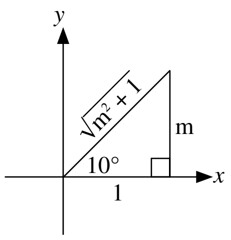
b) $\cos 155°$

$= -\cos 25°$

$= -\sqrt{1-p^2}$



8) $\tan 10° = \frac{m}{1}$



a) $\sin 10° = \frac{m}{\sqrt{m^2+1}}$

b) $\cos 10° = \frac{1}{\sqrt{m^2+1}}$

c) $\tan 90° = \frac{\sin 1\,080°}{80°}$

$= \frac{\frac{1}{\sqrt{m^2+1}}}{\frac{m}{\sqrt{m^2+1}}}$

$= \frac{1}{m}$

$x = 1$ $y = m$ $r^2 = m^2 + 1$ $r = \sqrt{m^2+1}$

9) a) LHS $\frac{2\sin 10°}{\sin 10°} = 2$

b) LHS $\frac{\cos 20°}{2\sin 70°} = \frac{\sin 70°}{2\sin 70°} = \frac{1}{2}$

c) $\frac{(\sin 70°)(-\cos 5°)}{(\cos 20°)(-\cos 5°)} = 1$

$\sin 70°$

d) $\frac{\cos^2 20° + \sin^2 20°}{\sin^2 40°(1 + \frac{\sin^2 50°}{\cos^2 50°})}$

Lessons 16–18

Activity 1

1. $\theta = \pm 55{,}5° + k\cdot 180°\ k \in \mathbb{Z}$
2. $\theta = -194{,}5° = k\cdot 360°$ or $\theta = 94.5° + k\cdot 360°\ \ k \in \mathbb{Z}$

   $\theta \in \{-265{,}5°\ ;\ -194{,}5°\ ;\ 94{,}5°\ ;\ 165{,}5°\}$
3. $\theta = \pm 90° + k\cdot 360°\ \ k \in \mathbb{Z}$
4. $\theta = \pm 180° + k\cdot 360°\ \ k \in \mathbb{Z}$
5. $\theta = 7° + k\cdot 90$ or $\theta = 76° + k\cdot 180°\ \ k \in \mathbb{Z}$

   $\theta \in \{-173°\ ;\ -104°\ ;\ -83°\ ;\ 7°\ ;\ 76°\ ;\ 97°\}$
6. $\theta = \pm 30° + k\cdot 180°\ ;\ k \in \mathbb{Z}$

7. $\cos 2\theta = \cos(90° - \theta + 40°)$

$2\theta = \pm(130° - \theta) + k \cdot 360°$; $k \in \mathbb{Z}$

$3\theta = 130° + k \cdot 360°$ or $\theta = -130° + k \cdot 360°$

$\theta = 43{,}3° + k\ 180°$ or $\theta = -130° + k \cdot 360°$ $k \in \mathbb{Z}$

Activity 2

1. $\theta = -10° + k \cdot 60°$
   $k \in \mathbb{Z}$ $\theta \in \{-10° ; 50°\}$
2. $\theta = 55° + k \cdot 90°$
   $k \in \mathbb{Z}$ $\theta \in \{-125° ; -35° ; 55° ; 145°\}$
3. $\theta = -40° + k \cdot 180°$
   $k \in \mathbb{Z}$ $\theta \in \{-220° ; -40°\}$
4. $\theta = 63{,}4° + \text{k}\ 180°$
   $k \in \mathbb{Z}$ $\theta \in \{-296{,}6° ; -116{,}6° ; 63{,}4° ; 243{,}4°\}$
5. $\theta = -26{,}6° + k \cdot 180°$
   $k \in \mathbb{Z}$ $\theta \in \{-26{,}6° ; 153{,}4°\}$
6. $\theta = -71{,}6° + k \cdot 180°$
   $k \in \mathbb{Z}$ $\theta \in \{-71{,}6° ; 108{,}4°\}$

Activity 3

1. $\tan\theta = \frac{1}{\sqrt{3}}$

   $\therefore \theta = -30° + k \cdot 180°$; $k \in \mathbb{Z}$

2. $\cos(90° - 3\theta) = \cos(\theta + 62°)$

   $\therefore 90° - 3\theta = \theta + 62° + k \cdot 360°$ or $90° - 3\theta = -\theta - 62° + k \cdot 360°$

   $\therefore -4\theta = -28° + k \cdot 360°$ or $-2\theta = -152° + k \cdot 360°$

   $\therefore \theta = 7° - k \cdot 90$ or $\theta = 76° - k \cdot 180°$ $k \in \mathbb{Z}$

3. $\cos 2\theta = -0{,}357$

   $2\theta = \pm 110{,}92° + k \cdot 360°$; $k \in \mathbb{Z}$

   $\theta = \pm 55{,}5° + k \cdot 180°$

4. $\sin 2\theta = -1$

   $\theta = -90° + k \cdot 360$ or $2\theta = 180° - (-90°) + k \cdot 360°$

   $\theta = -45° + k \cdot 180°$ $k \in \mathbb{Z}$

5. $\sin(3\theta + 24°) = -0{,}279$

   $3\theta + 24° = -16{,}2° + k \cdot 360°$ or $3\theta + 24° = 180° - (-16{,}2°) + k \cdot 360°$

   $3\theta = -40{,}2° + k \cdot 360°$ or $3\theta = -172{,}2° + k \cdot 360°$

   $\theta = -13{,}4° + k \cdot 120°$ or $\theta = 57{,}4° + k \cdot 120°$ $k \in \mathbb{Z}$

6. $\cos(\theta + 50°) = -0{,}814$

$\theta + 50° = 144{,}49° + k\cdot 360°$ or $\theta + 50° = -144{,}49° + k\cdot 360°$ $k \in \mathbb{Z}$

$\theta = 94{,}49° + k\cdot 360°$ or $\theta = -194{,}49° + k\cdot 360°$ $k \in \mathbb{Z}$

7. $\tan^2 3\theta = 3$

$\therefore \tan 3\theta = \pm\sqrt{3}$

$\therefore 3\theta = \pm 60° + k\cdot 180°$

$\therefore \theta = \pm 20° + k\cdot 60°$ $k \in \mathbb{Z}$

8. $\cos 2x = \cos(90° - x + 40°)$

$\therefore \cos 2x = \cos(130° - x)$

$\therefore 2x = \pm(130° - x) + k\cdot 360°$

$\therefore 2x = 130° - x + k\cdot 360°$ or $2x = -130° + x + k\cdot 360°$

$\therefore 3x = 130° + k\cdot 360°$ or $x = -130° + k\cdot 360°$

$\therefore x = 43{,}3° + k\cdot 120°$ or $x = -130° + k\cdot 360°$ $k \in \mathbb{Z}$

9. $\tan(90° - x) = \tan(2x + 60°)$

$\therefore 90° - x = 2x + 60° + k\cdot 180°$

$\therefore -3x = -30° + k\cdot 180°$

$\therefore x = 10° + k\cdot 60°$ $k \in \mathbb{Z}$

10. $\cos\theta = \frac{-1}{\sqrt{2}}$

$\therefore \theta = \pm 135° + k\cdot 360°$ $k \in \mathbb{Z}$

11. $\sin 2\theta = -\frac{1}{\sqrt{2}}$

$2\theta = -45° + k\cdot 360°$ or $2\theta = 180° - (-45°) + k\cdot 360°$

$\therefore \theta = -22{,}5° + k\cdot 180°$ or $\theta = 112{,}5° + k\cdot 180°$ $k \in \mathbb{Z}$

Activity 4

1. $2\sin^2\theta - 3\cos^2\theta = 2(\sin^2\theta + \cos^2\theta)$

$2\sin^2\theta - 3\cos^2\theta = 2\sin^2\theta + 2\cos^2\theta$

$0 = 2\cos^2\theta + 3\cos^2\theta$

$\cos^2 = 0$

$\theta = \pm 90° + k\, 360°$ $k \in \mathbb{Z}$

$\{\pm 90°\ ;\ \pm 270°\}$

2. $2\sin\theta\cos\theta - \sin\theta + 2\cos\theta - 1 = 0$

$\sin\theta(2\cos\theta - 1) + (2\cos\theta - 1) = 0$

$(2\cos\theta - 1)(\sin\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\sin\theta = -1$

$\theta = \pm 60° + k\cdot 360°$ or $\theta = -90° + k\cdot 360°$ $k \in \mathbb{Z}$

3. $7\sin^2\theta + 4\sin\theta\ \cos\theta\ = 3\sin^2\theta + 3\cos^2\theta$

$4\sin^2\theta + 4\sin\theta\cos\theta\ - 3\cos^2\theta\ = 0$

$(2\sin\theta - \cos\theta)(2\sin\theta + 3\cos\theta) = 0$

$2\sin\theta = \cos\theta$ or $2\sin\theta = -3\cos\theta$

$\tan\theta\ = \frac{1}{2}$ or $\tan\theta = -\frac{3}{2}$

$\theta = 26{,}6° + k\cdot180°$ or $\theta = -56{,}3° + k\cdot180°\ k \in \mathbb{Z}$

$\{26{,}6°\ ; -153{,}4°\ ; -56{,}3°\ ; 123{,}7°\ ; 206{,}6°\ ; 303{,}7°\}$

4. $\cos(5\theta - 40°) = \cos(90° - 2\theta + 20°)$

$\cos(5\theta - 40°) = \cos(110° - 2\theta)$

$5\theta - 40° = 110° - 2\theta + k\cdot360°$ or $5\theta - 40° = 2\theta - 110° + k\cdot360°$

$7\theta = 150° + k\cdot360$ or $3\theta = -70° + k\cdot360°$

$\theta = \frac{150°}{7} + \frac{k360°}{7}$ or $\theta = -23{,}3° + k\cdot120°\ k \in \mathbb{Z}$

5. $8\cos x = 4(1 - \cos^2) - 7$

$8\cos x = 4 - 4\cos^2 x - 7$

$4\cos^2 x + 8\cos x + 3 = 0$

$(2\cos x + 1)(2\cos x + 3) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x = -\frac{3}{2}$

invalid

$x = \pm120° + k\cdot360°$

$\{-120°\ ; -240°\}$

6. $5\sin^2\theta + 2\cos\theta - 5\cos\theta\sin\theta - 2\sin\theta = 0$

$5\sin\theta(\sin\theta - \cos\theta) + 2(\cos\theta - \sin\theta) = 0$

$(\sin\theta - \cos\theta)(5\sin\theta - 2) = 0$

$\sin\theta = \cos\theta$ or $\sin\theta = \frac{2}{5}$

$\tan\theta = 1$

$\theta = 45° + k\cdot180°$ or $\theta = 23{,}6° + k\cdot360°$

or $\theta = 156{,}4° + k\cdot360°$

7. $\sin 2A = \frac{\sin 2A}{\cos 2A}$

$\sin 2A\cos 2A = \sin 2A$

$\sin 2A\cos 2A - \sin 2A = 0$

$\sin 2A(\cos 2A - 1) = 0$

$\sin 2A = 0$ or $\cos 2A = 1$

$2A = 0 + k\cdot360°$ or $2A = 180° + k\cdot360°$

$A = 0 + k\cdot180°$ or $A = 90° + k\cdot180°\quad k \in \mathbb{Z}$

$\{0\ ; 180°\ ; -180°\ ; 90°\ ; -90°\}$

8. $4\sin^2 x + 2(1 - \sin^2 x) = 5 \text{ in } x$

$4\sin^2 x + 2 - 2\sin^2 x - 5\sin x = 0$

$2\sin^2 x - 5\sin x + 2 = 0$

$(2\sin x - 1)(\sin x - 2) = 0$

$\sin x = 1/2$ or $\sin x = 2$

Invalid

$x = 30° + k\cdot 360°$ or $x = 150° + k\cdot 360°$ $k \in \mathbb{Z}$

Activity 5

1. $\cos 2A = \cos(180° - 3A)$

$2A = 180° \pm 3A + k\cdot 360°$

$5A = 180° + k\cdot 360°$ or $-A = 180° + k\cdot 360°$

$A = 36° + k\cdot 72°$ or $A = -180° + k\cdot 360°,$ $k \in \mathbb{Z}$

2. $2(1 - \sin^2 x) = 3\sin x + 3$

$2 - 2\sin^2 x = 3\sin x + 3$

$2\sin^2 x + 3\sin x + 1 = 0$

$(2\sin x + 1)(\sin x + 1) = 0$

$\sin x = -\frac{1}{2}$ or $\sin x = -1$

$x = -30° + k\cdot 360°$ or $x = 210° + k\cdot 360°$ or $x = -90° + k\cdot 360°$

$\{-150° ; -90° ; -30° ; 210° ; 270° ; 330°\}$

3. $10\cos^2\theta - 10\sin\theta\cos\theta + 2\sin^2\theta + 2\cos^2\theta = 0$

$12\cos^2\theta - 10\sin\theta\cos\theta + 2\sin^2\theta = 0$

$6\cos^2\theta - 5\sin\theta\cos\theta + \sin^2\theta = 0$

$(3\cos\theta - \sin\theta)(2\cos\theta - \sin\theta) = 0$

$3\cos\theta = \sin\theta$ or $2\cos\theta = \sin\theta$

$\tan\theta = 3$ or $\tan\theta = 2$

$\theta = 71{,}6°$ or $\theta = 63{,}4° + k\cdot 180°$ $k \in \mathbb{Z}$

4. $\cos(\theta - 30°) = \cos 40°$

$\theta - 30° = 40° + k\cdot 360°$ or $\theta - 30° = -40° + k\cdot 360°$

$\theta = 70° + k\cdot 360°$ or $\theta = -10° + k\cdot 360°$ $k \in \mathbb{Z}$

5. $\cos 3x = \sin(-4x)$

$\cos 3x = \cos(90° \pm 4x)$

$\therefore 3x = 90° \pm 4x + k\cdot 360°$

$\therefore 3x = 90° + 4x + k\cdot 360°$

$\therefore -x = 90° + k\cdot 360°$

$\therefore x = -90° + k\cdot 360°$ $k \in \mathbb{Z}$

or

$3x = 90° - 4x + k \cdot 360°$

$7x = 90° + k \cdot 360°$

$x = 12{,}9° + k \cdot 51{,}4°$

6. $3 \sin^2 x = 5 \cos^2 x$

$\tan^2 x = \frac{5}{3}$

$\tan x = \sqrt{\frac{5}{3}}$ or $\tan x = -\sqrt{\frac{5}{3}}$

$x = \pm 52{,}2 + k \cdot 180° \quad k \in \mathbb{Z}$

7. $(1 - \sin^2 x) - \sin^2 x - 3 \sin x - 2 = 0$

$-2 \sin^2 x - 3 \sin x - 1 = 0$

$2\sin^2 x + 3 \sin x + 1 = 0$

$(2 \sin x + 1)(\sin x + 1) = 0$

$\sin x = -\frac{1}{2}$ or $\sin x = -1$

$x = -30° + k\,360°$ or $x = 210° + k \cdot 360°$ or $x = -90° + k \cdot 360° \quad k \in \mathbb{Z}$

8. $\cos^2\theta - 2 \sin\theta \cos\theta - (\sin^2\theta + \cos^2\theta) = 0$

$2 \sin\theta \cos\theta + \sin^2\theta = 0$

$\sin\theta\,(2 \cos\theta + \sin\theta) = 0$

$\sin\theta = 0$ or $2 \cos\theta = -\sin\theta$

$\theta = 0° + k \cdot 360°$ or $-2 = \tan\theta$

$\theta = -63{,}4° + k \cdot 180°$ or $\theta = 180° + k \cdot 360° \quad k \in \mathbb{Z}$

9. $2 \sin^2 x - \cos x + 2 \sin x \cos x = \sin x$

$2\sin^2 x + 2 \sin x \cos x - \cos x - \sin x = 0$

$2\sin x(\sin x + \cos x) - (\cos x + \sin x) = 0$

$(\sin x + \cos x)(2 \sin x - 1) = 0$

$\sin x = -\cos x$ or $\sin x = \frac{1}{2}$

$\tan x = -1$

$x = -45° + k \cdot 180°$ or $x = 30° + k \cdot 360°$ or $x = 150° + k \cdot 360°$

$\{-45° ; 30° ; 135° ; 150°\}$

Lesson 19

Activity 1

(a) 4; 7; 10; 13; ……………………………..

$T_n = 4 + (n - 1)3$

$\therefore T_n = 4 + 3n - 3$

$\therefore T_n = 3n + 1$

$T_n = 3n + 1$

$\therefore T_{150} = 3(150) + 1 = 301$

(b) 0; 4; 8; 12; ………………………………

$T_n = 0 + (n - 1)4$

$\therefore T_n = 0 + 4n - 4$

$\therefore T_n = 4n - 4$

$T_n = 4n - 4$

$\therefore T_{150} = 4(150) - 4 = 596$

(c) 10; 6; 2; ………………………………..

$T_n = 10 + (n - 1)(-4)$

$\therefore T_n = 10 - 4n + 4$

$\therefore T_n = -4n + 14$

$T_n = -4n + 14$

$\therefore T_{150} = -4(150) + 14 = -586$

Activity 2

1. (a) 3; 12; 27; 48; ……………………………..

3 12 27 48 75

9 15 21 27

6 6 5

$2a = 6$ | $3a + b = 9$ | $a + b + c = 3$

$\therefore a = 3$ | $\therefore 3(3) + b = 9$ | $\therefore 3 + 0 + c = 3$

$\therefore b = 0$ | $\therefore c = 0$

$\therefore T_n = 3n^2$

(b) 7; 24; 51; 88; ……………………………

7 24 51 88 135

17 27 37 47

10 10 10

$2a = 10$ | $3a + b = 17$ | $a + b + c = 7$

$\therefore a = 5$ | $\therefore 3(5) + b = 17$ | $\therefore 5 + 2 + c = 7$

$\therefore b = 2$ | $\therefore c = 0$

$\therefore T_n = 5n^2 + 2n$

(c) 0; 3; 8; 15; ………………………………..

0 3 8 15 24

3 5 7 9

2 2 2

$2a = 2$	$3a + b = 3$	$a + b + c = 0$
$\therefore a = 1$	$\therefore 3(1) + b = 3$	$\therefore 1 + 0 + c = 0$
	$\therefore b = 0$	$\therefore c = -1$

$\therefore T_n = n^2 - 1$

(d) 6; 17; 34; 57; …………………….

6 17 34 57

11 17 23 $\longrightarrow$ First difference

6 6 $\longrightarrow$ Second difference

$2a = 6$	$3a + b = 11$	$a + b + c = 6$
$\therefore a = 3$	$\therefore 3(3) + b = 11$	$\therefore 3 + 2 + c = 6$
	$\therefore b = 2$	$\therefore c = 1$

$\therefore T_n = 3n^2 + 2n + 1$

2. (a) 1; 3; 6; 10; 15; ……………………….

1 3 6 10

2 3 4 $\longrightarrow$ First difference

1 1 $\longrightarrow$ Second difference

$2a = 1$	$3a + b = 2$	$a + b + c = 1$
$\therefore a = \frac{1}{2}$	$\therefore 3\left(\frac{1}{2}\right) + b = 2$	$\therefore \frac{1}{2} + \frac{1}{2} + c = 1$
	$\therefore b = \frac{1}{2}$	$\therefore c = 0$

$S_n = \frac{1}{2}n^2 + \frac{1}{2}n$

(b) $S_{200} = \frac{1}{2}(200)^2 + \frac{1}{2}(200) = 20\ 100$

Lesson 20

Activity 1

(a) 4; 7; 10; 13; ……………………………..

$T_n = T_n - 1 + 3$

(b) 0; 4; 8; 12; ………………………………

$T_n = T_{n-1} + 4$

(c) 10; 6; 2; ………………………………..

$T_n = T_{n-1} - 4$

Activity 2

1. (a) 7; 24; 51; 88; ………………………

$T_n = T_{n-1} + 10(n-1) + 7$

$\therefore T_{n-1} + (10n - 3)$

(b) 3; 12; 27; 48; ……………………….

$T_n = T_{n-1} + 6(n-1) + 3$

$\therefore T_n = T_{n-1} + (6n - 3)$

(c) 6; 17; 34; 57; …………………….

$T_n = T_{n-1} + 6(n-1) + 5$

$\therefore T_n = T_{n-1} + (6n - 1)$

Lesson 21

Activity 1

1. a) $T_n = 4\left(-\frac{1}{2}\right)^{n-1}$ b) $T_n = -\frac{1}{8}(4)^{n-1}$

c) $T_n = 32(2)^{n-1}$ d) $T_n = 3a(2a)^{n-1}$

2. a) $T_n = \left(\frac{1}{3}\right)^{n-1}$ b) $T_8 = \left(\frac{1}{3}\right)^7 = \frac{1}{243}$

3. a) $T_n = \left(\frac{3}{2}\right)^{n-1}$ b) $\left(\frac{3}{2}\right)^{n-1} = \frac{243}{32}$

$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^5$

$n = 6$

4. $120;\ 120\left(\frac{9}{10}\right);\ 120\left(\frac{9}{10}\right)^2$

$T_8 = 120\left(\frac{9}{10}\right)^7$

$= 0{,}3$ cm

Activity 2

Rubric

Topic	1	2
1. Historical facts	Very few supplied	
2. The Fibonacci		

Activity 3

1. a)

× × × × ×
● × × × ×
● ● × × × Add the same number of crosses to form a rectangle 5×6
● ● ● × ×
● ● ● ● ×
● ● ● ● ●

$\therefore$ Dots $\frac{5 \times 6}{2}$

b) $\frac{n(n+1)}{2}$

2. a)

●
● ●
● ● ●

● × × ×
● ● × ×
● ● ● ×
● ● ● ●

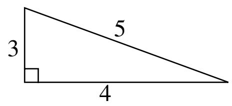
put T_3 or T_4 you get S_4

$\therefore S_4 = T_3 + T_4$

b) $S_n = T_{n-1} + T_n$

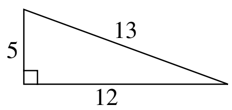
3. a) 1;4; 9;16; 25; 36; 49; 64; 81; 100

b) $9 + 16 = 25$



c) $6^2 + 8^2 = 10^2$

$= 36 + 64 = 100$



$9^2 + 12^2 = 15^2$

$81 + 144 = 225$

$25 + 144 = 169$

4. Square numbers can only end in

1; 4;9; 6;5

$\therefore$ c is not a square number

$(a + b)^2 = a^2 + 2(a)(b) + b^2$

a) 441

$= 400 + 40 + 1$

$= (20)^2 + 2(20)(1) + 1^2$

$= (20 + 1)^2$ square number

b) If 2001 is a square number then the number must end in 1 or 9.

$40^2 = 1600$ $50^2 = 2500$

$41^2 = (40 + 1)^2$

$= 1600 + 80 + 1 \neq 2001$

$49^2 = (40 + 9)^2$

$= 1600 + 98 + 81$

$= 1779$

$\therefore$ 2001 is not a perfect square

c) No

d) 4096

$(60)^2 = 3600$ $(70)^2 = 4900$

Between 60 and 70 to end in 6 must be 64 or 66.

$64(2) = (40 + 4)2$

$= 3600 + 480 + 16$

4096

$\therefore$4096 is a square number

5. Let the numbers be x and $x + 1$

$\therefore (x + 1)2 - x^2$

$= x^2 + 2x + 1 - x^2$

$= 2x + 1$

$\therefore 2x$ is even $\therefore 2x + 1$ is odd

6. a) $S_7 - S_4$

$= 49 - 16$

$= 33$ m of 3

c) $S_8 - S_5$

$= 64 - 25 = 39$ m of 3

d) $\mathrm{S}_{(x+y)} - \mathrm{S}_x$ is a multiple of y.

e) $64 = (2^2)^3$

$664 = (2^3)^2$

So

$(3^2)^3 = (9)^3 = 729$

$(3^3)^2 = (27)^2 = 729$

$(4^2)^3 = (16)^3 = 4096$

$(4^3)^2 = (64)^2 = 4096$

Generalisation

$(x^2)^3 = (x^3)^2$

7. a) Factors of a prime number (p) are p and 1.

$\therefore t(p) = 2$

b) Factors of 2^n are

1; 2; 2^2; … 2^n

$\therefore$ $(n + 1)$ factors

$\therefore$ $t(p^n) = (n + 1)$

c) $t(8 \times 4) = 6$

$t(80 \times t(4) = 4 \times 3 = 12$

$t(8 \times 4) \neq t(8) \times t(4)$

$t(5 \times 6) = 8$

$t(5) \times \text{t}(6) = 2 \times 4 = 8$

$t(5 \times 6) = t(5) \times t(6)$

(This only occurs when the factors are consecutive counting numbers – co prime)

8. Your investigation. What did you find.

9. $1^3 + 2^3 + \ldots n^3 = (1 + 2 + \ldots + n)^2$

10. a) $\text{LD}(n) \times \text{LD}(m)$

b) 1;4; 5;6; 9

c) ends in 7

11. a) 20; 24; 30 (The next prime numbers plus 1)

b) 14; 4;4 (The product of each number)

c) Pentagonal numbers

1;5; 12; 22 …

4;7; 10

3 3

$$T_n = \tfrac{3}{2}n^2 + an + b$$

$$1 = \tfrac{3}{2} + a + b$$

$$2 = 3 + 2a + 2b$$

$$2a + 2b = -1$$

$$5 = 6 + 2a + b$$

$$2a + b = -1$$

$$\underline{2a + 2b = -1}$$

$$-b = 0$$

$$b = a$$

$$2a = -1 - 0$$

$$\text{A} = -\tfrac{1}{2}$$

$$T_n = \tfrac{3}{2}n^2 - \tfrac{1}{2}$$

Or

1 5 12 22 35

4 7 10 13

3 3 3

	T_1	T_2	T_3	T_4
$\frac{3}{2}n^2$	$\frac{3}{2}$	6	$\frac{27}{2}$	24
want	1	5	12	22
short	$-\frac{1}{2}$	$-\frac{2}{2}$	$-\frac{3}{2}$	$-\frac{4}{2}$

$$\text{So } T_n = \tfrac{3}{2}n^2 - \tfrac{n}{2}$$

$$= \tfrac{n}{2}(3n - 1)$$

Lesson 22

Activity 2

1. a) $\frac{\sin \hat{C}}{3} = \frac{\sin 120°}{4}$

$\therefore \sin \hat{C} = 0,6495$

$\hat{C} = 40,5°$

$\therefore \hat{B} = 19,5°$

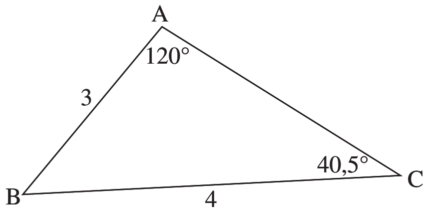
so area $= \frac{1}{2}(3)(4) \sin 19,5°$

$= 2\ u^2$

also

$\frac{AC}{\sin B} = \frac{4}{\sin 120°}$

$\therefore AC = \frac{4 \sin 19,5}{\sin 120°} = 1,54$



2. a) Area $\Delta XYZ = \frac{1}{2}(3)(3) \sin 102°$

$= 4,4\ u^2$

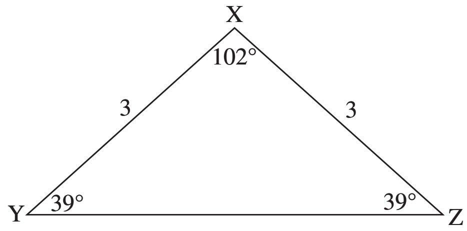
Further $\hat{Y} = \hat{Z} = \frac{180 - 102°}{2}$

$= 39°$

Now area $\Delta XYZ = \frac{1}{2}(3)(YZ) \sin 39°$

$\therefore 4,4 = \frac{3}{2}\ YZ \sin 39°$

$\therefore YZ = 4,66$



b) Volume = base area × height

$= 4,4 \times 12$

$= 52,8\ u^3$

$\hat{Q} = 180° - (151° + 12°)$

$\hat{Q} = 17°$

Now $\frac{PR}{\sin 17°} = \frac{3}{\sin 12°}$

$\therefore PR = 4,22$

and $\frac{PQ}{\sin 151} = \frac{3}{\sin 12°}$

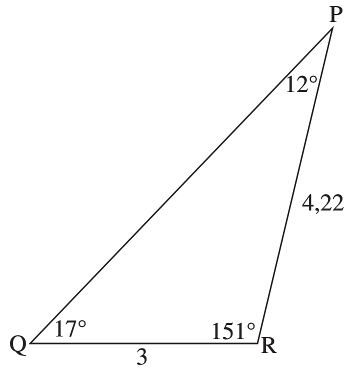
$\therefore PQ = 6,995 \approx 7$

Lastly area $\Delta PQR = \frac{1}{2}(7)(3) \sin 17°$

$= 3,07\ u^2$

alternatively: area $\Delta PQR = \frac{1}{2}(4,22)(3) \sin 151$

$= 3,07\ u^2$



3. $B_1 = 180° - 81° = 99°$

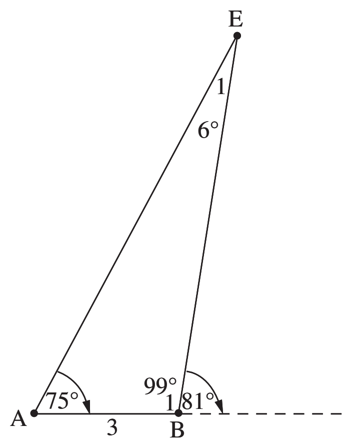
Now $\hat{E}_1 = 180° - 99° - 75°$

$\hat{E}_1 = 6°$

So: $\frac{EB}{\sin 75°} = \frac{3}{\sin 6°}$

$\therefore EB = \frac{3 \sin 75°}{\sin 6°}$

$EB = 27{,}72$ m



4. a) $\frac{\sin \hat{A}}{5{,}3} = \frac{\sin 42°}{5{,}9}$

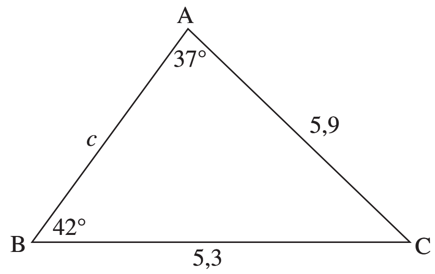
$\therefore \sin \hat{A} = 0{,}601$

$\hat{A} = 36{,}95° \approx 37°$

$\therefore \hat{C} = 101°$

$\therefore \frac{AB}{\sin 101} = \frac{5{,}3}{\sin 37°}$

$\therefore AB = 8{,}65$



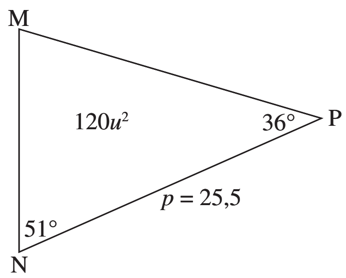
b) Area $= \frac{1}{2}p.(16) \sin 36° = 120$

$\therefore p = 25{,}5$

$\hat{M} = 93°$

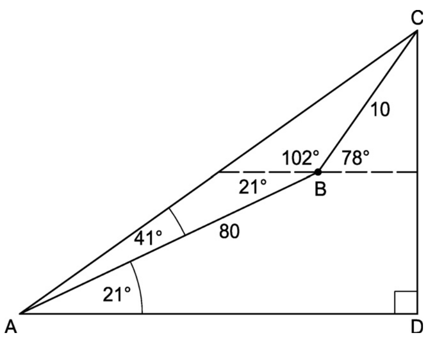
$\therefore \frac{16}{\sin 93°} = \frac{MN}{\sin 36°}$

$MN = 9{,}4$



Activity 3

1.



$\frac{AC}{\sin 123} = \frac{80}{\sin 16}$

$\therefore \frac{80 \sin 123}{\sin 16}$

$= 243{,}4$

$\frac{CD}{\sin 62°} = \frac{243{,}4}{\sin 90°}$

$\therefore$ BD

$= \frac{10 \sin 43°}{\sin 310°}$

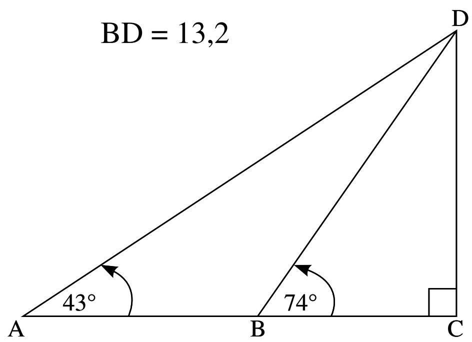
$BD = 13{,}2$

2. $D = 31°$

$\frac{BD}{\sin 43} = \frac{10}{\sin 31}$ $\therefore BD = \frac{10 \sin 43°}{\sin 310}$

$BD = 13{,}2$

$\frac{DC}{\sin 74°} = \frac{13{,}2}{\sin 90°}$ $DC = 12{,}7$ m



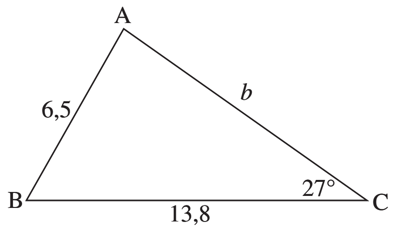
Lesson 23

Activity 4

1. a) $\dfrac{\sin \hat{A}}{13,8} = \dfrac{\sin 27^\circ}{6,5}$

$\therefore \sin \hat{A} = 0,96^\circ$

$\hat{A} = 74,5^\circ$ or $\hat{A} = 105,5^\circ$

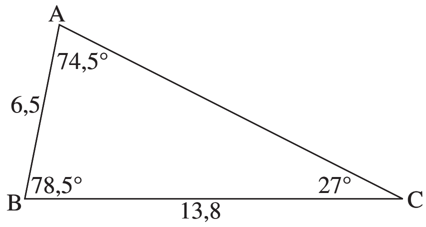


If $\hat{A} = 74,5^\circ$

$\hat{B} = 180^\circ - 74,5^\circ - 27^\circ = 78,5^\circ$

$\therefore \dfrac{AC}{\sin 78,5^\circ} = \dfrac{6,5}{\sin 27^\circ}$

$\therefore AC = 14$

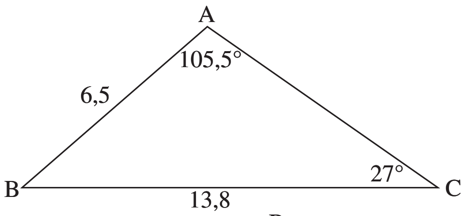


If $\hat{A} = 105,5^\circ$

$\hat{B} = 180^\circ - 105,5^\circ - 27^\circ = 47,5^\circ$

$\therefore \dfrac{AC}{\sin 47,5} = \dfrac{6,5}{\sin 27}$

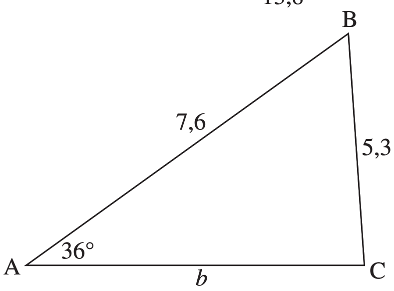
$AC = 10,6$



b) $\dfrac{\sin \hat{C}}{7,6} = \dfrac{\sin 36}{5,3}$

$\sin \hat{C} = 0,842$

$\hat{C} = 57,4^\circ$ or $\hat{C} = 122,6^\circ$



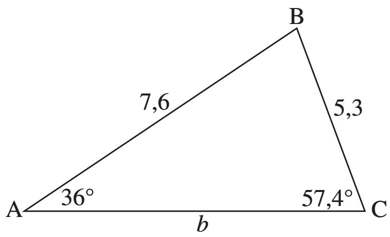
If $\hat{C} = 57,4^\circ$:

$\hat{B} = 180^\circ - 36^\circ - 57,4^\circ$

$= 86,6^\circ$

$\therefore \dfrac{AC}{\sin 86,6^\circ} = \dfrac{7,6}{\sin 57,4^\circ}$

$\therefore AC = 9$

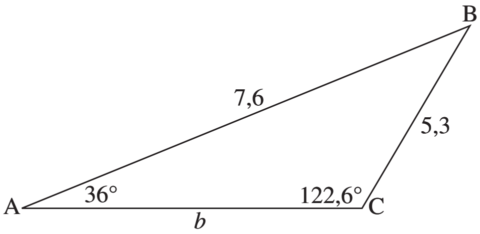


If $\hat{C} = 122,6^\circ$

$\hat{B} = 21,4^\circ$

$\dfrac{AC}{\sin 21,4^\circ} = \dfrac{7,6}{\sin 122,6}$

$AC = 3,3$



c) $\frac{\sin \hat{C}}{40} = \frac{\sin 36°}{30}$

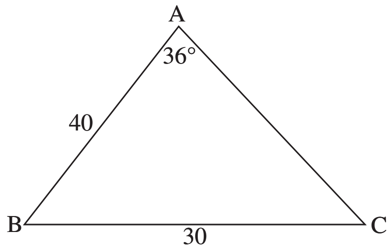
$\sin \hat{C} = 0{,}7837$

$\hat{C} = 51{,}6°$ or $\hat{C} = 128{,}4°$

If $\hat{C} = 128{,}4°$ $\quad\quad \hat{B} = 15{,}6°$

$\frac{AC}{\sin 15{,}6°} = \frac{30}{\sin 36°}$

$AC = 13{,}7$



2. $Q\hat{P}S = 180° - (75° + 8°) = 97°$

Now PS: (In ΔPQS)

$\frac{PS}{\sin 75°} = \frac{800}{\sin 97°}$

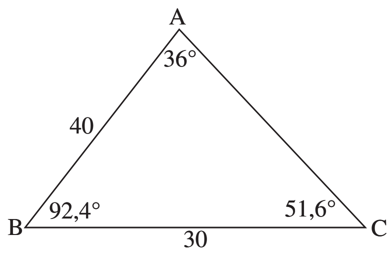
$\therefore PS = 778{,}5$ m

In ΔPRS: ($\hat{R} = 90°$)

$\therefore \sin 8° = \frac{h}{PS}$

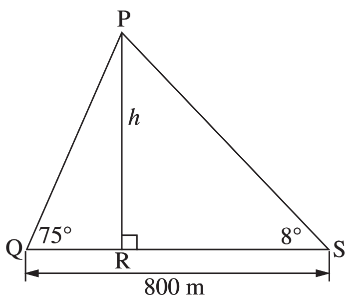
$\therefore h = 778{,}5 \times \sin 8$

$= 108{,}3$ m



3. $\frac{x}{\sin 62} = \frac{20}{\sin 81}$

$x = \frac{20 \sin 62}{\sin 81} = 17{,}8$ m



4. a) $4^2 = 2^2 + 3^2 - 2(2)(3) \cos \hat{E}$

$\therefore \cos \hat{E} = \frac{2^2 + 3^2 - 4^2}{4.3} = -0{,}25$

$\therefore \hat{E} = 104{,}5°$

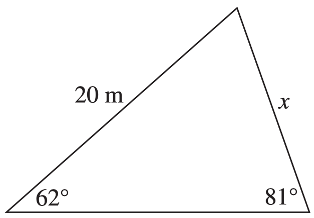
and $\frac{4}{\sin 104{,}5°} = \frac{2}{\sin F}$

$\therefore \sin F = 0{,}48$

$\therefore \hat{F} = 29°$

$\therefore D = 180° - 29° - 104{,}5$

$= 46{,}5°$



b) $RQ = \sqrt{(2{,}2)^2 + (3{,}9)^2 - 2(2{,}2)(3{,}9) \cos 49}$

$= 2{,}96$

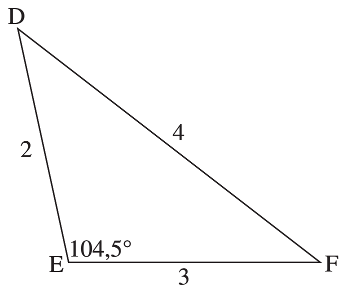
$\frac{\sin \hat{Q}}{2{,}2} = \frac{\sin 49}{2{,}96}$

$\sin \hat{Q} = 0{,}56$

$\therefore \hat{Q} = 34{,}1°$

Then $\hat{R} = 180° - 34{,}1° - 49°$

$\hat{R} = 96{,}9°$



c) $BC = \sqrt{9^2 + 8^2 - 2(9)(8)\cos 60}$

$= 8{,}54$

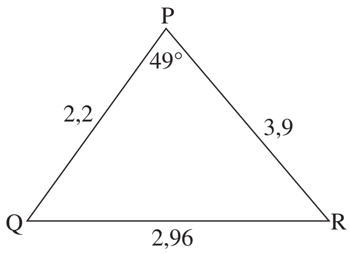
so $\frac{\sin \hat{C}}{9} = \frac{\sin 60}{8{,}54}$

$\therefore \sin \hat{C} = 0{,}9126$

$\hat{C} = 65{,}9°$

Then $B = 180° - 65{,}9° - 60°$

$B = 54{,}1°$



Activity 5

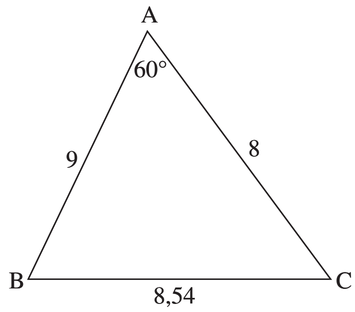
1. a) $AC = 33$ cm

$\cos \hat{B} = \frac{19^2 + 25^2 - 33^2}{2(19)(25)} = -0{,}108$

$\therefore \hat{B} = 96{,}2° = \hat{D}$ (opp Δs equal)

Then $A = C = (180° - 96{,}2°)$ co-int Δs)

$= 83{,}8°$

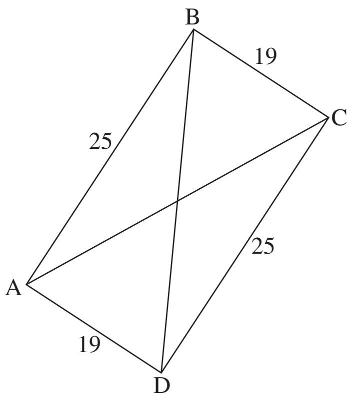


b) $BD = \sqrt{19^2 + 25^2 - 2(19)(25)\cos 83{,}8°}$

$= 29{,}7$ cm

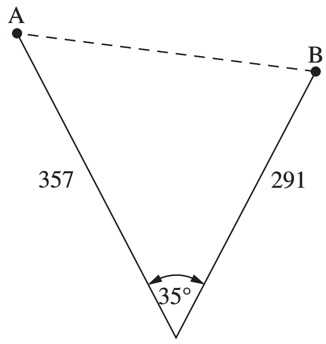
2. $AB = \sqrt{291^2 + 357^2 - 2(291)(357)\cos 35}$

$= 204{,}8$ m



3. a) Area $\Delta ABC = \frac{1}{2}(8{,}4)(5{,}1)\sin 55$

$= 17{,}55\ u^2$



b) Area ΔAIB + Area ΔBIC + Area ΔAIC

$= \frac{1}{2}AB \cdot r + \frac{1}{2}BC \cdot r + \frac{1}{2}AC \cdot r$

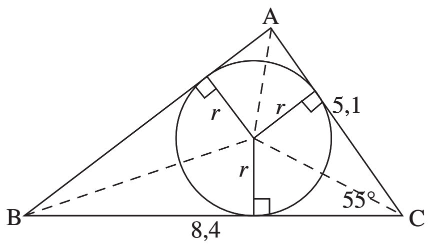
$= \frac{1}{2}r(AB + BC + AC)$

But $AB = R8{,}4^2 + 5{,}1^2 - 2(5{,}1)(8{,}4)\cos 55$

$= 6{,}89$

$\therefore 17{,}55 = \frac{1}{2}r(6{,}89 + 5{,}1 + 8{,}4)$

$\therefore r = 1{,}72$



Lesson 24

Activity 1

1. a) $DB = \sqrt{152^2 + 120^2 - 2(152)(120)\cos 130°}$

$= 79,5$

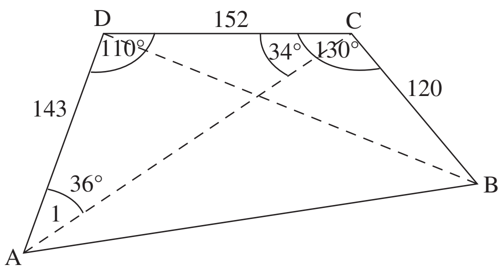
$AC = \sqrt{143^2 + 152^2 - 2(143)(152)\cos 110°}$

$= 241,7$

Now: $\frac{\sin A_1}{152} = \frac{\sin 110°}{241,7}$

$\therefore \sin A_1 = 0,591$

$\therefore \hat{A}_1 = 36°$



2. In ΔOPQ:

$P\hat{Q}O = 140°$

so $OP = \sqrt{400^2 + 50^2 - 2(50)(400)\cos 140}$

$= 439,5$ km/h

For the hearing:

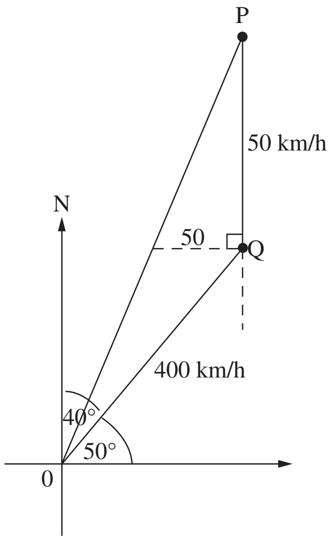
$\frac{\sin Ø}{50} = \frac{\sin 140}{439,5}$

$\therefore \sin Ø = \frac{50 \times \sin 140}{439,5}$

$\sin Ø = 0,073$

$Ø = 4,1934°$

So the bearing was $40° - 4,19° = 35,8°$



3.1. 30 km/h × 4h = 120 km

3.2 500 km/h × 4h = 2 000 km

3.3 If AB + BC = 2 000 km, then if AB = x, BC = 2 000 – x.

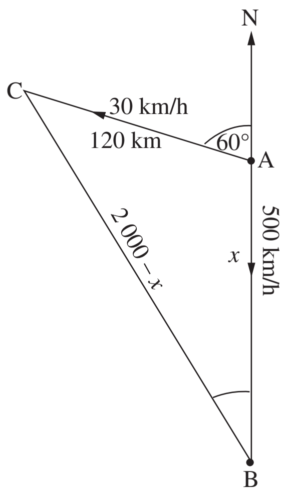
So:

$(2\,000 - x)^2 = 500^2 + x^2 - 2(500)(x)\cos 120°$

$4\,000\,000 - 4\,000x + x^2 = 250\,000 + x^2 + 500x$

$\therefore -4\,500x = -3\,750\,000$

$\therefore x = 833$ km



3.4 $\frac{\sin B}{120} = \frac{\sin 120}{2\,000 - 833}$

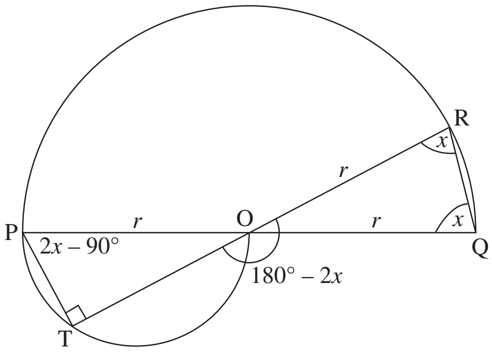
$\therefore \sin B = 0,08905$

$\hat{B} = 5{,}1°$

$\therefore$ Direction is N5°W or a bearing of 355°.

4. $\hat{P} = 90° - 180° + 2x$

$= 2x - 90°$



4.1 $\dfrac{PQ}{\sin(180 - 2x)} = \dfrac{r}{\sin x}$

$\therefore$ RQ $= r \sin x.\sin 2x$

$= 2r \sin^2 (180 - 2x)$

4.2 Area $= \frac{1}{2}r^2 \sin(180 - 2x)$

$= \frac{1}{2}r^2 \sin 2x$

4.3 $\sin(180° - 2x) = \frac{PT}{r}$

PT = r sin 2x

and $\cos 180° - 2x = \frac{OT}{r}$

$\therefore$ OT $= -r \cos 2x$ Thus area ΔPOT $= \frac{1}{2}$PT.OT

$= -\frac{1}{2}r^2 \sin 2x \cos 2x$

4.4 $\dfrac{\text{Area } \Delta ROQ}{\text{Area } \Delta POT}$

$= \dfrac{\frac{1}{2}r^2 \sin 2x}{-\frac{1}{2}r^2 \sin 2x \cos 2x}$

$= \dfrac{-1}{\cos 150°}$

$= \dfrac{-1}{\frac{-\sqrt{3}}{2}}$

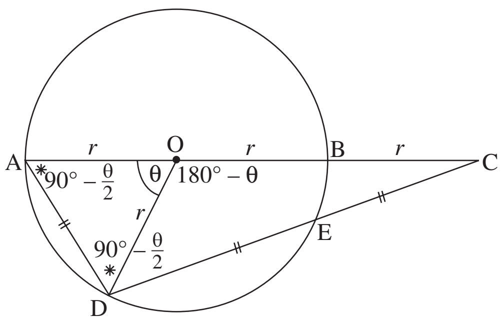
$= \dfrac{2}{\sqrt{3}}$

5.1 In ΔAOD

$AD^2 = r^2 + r^2 - 2r^2 \cos\theta$

$= 2r^2 - 2r^2 \cos\theta$

$= 2r^2(1 - \cos\theta)$ …(1)



In ΔDOC: DC = 2AD $\Rightarrow DC^2 = 4AD^2$

$\therefore 4AD^2 = r^2 + (2r)^2 - 2(2r)(r)\cos(180 - \theta)$

$= 5r^2 + 4r^2 \cos\theta$

$\therefore AD^2 = \frac{1}{4}(5r^2 + 4r^2 \cos\theta)$…(2)

Now: (1) = (2)

$2r^2(1 - \cos\theta) = \frac{1}{4}(5r^2 + 4r^2\cos\theta)$

$8r^2 - 8r^2\cos\theta = 5r^2 + 4r^2\cos\theta$

$3r^2 = 12r^2\cos\theta$

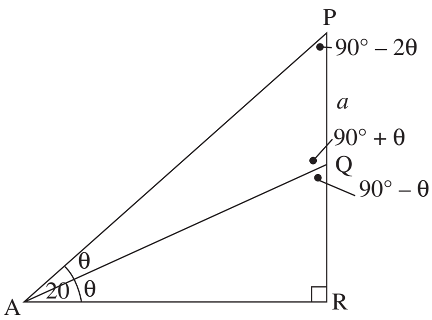
$\therefore \cos\theta = \frac{1}{4}$

5.2 Area $\Delta AOD = \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2\frac{\sqrt{15}}{4}$

$= \frac{r^2\sqrt{15}}{8}$

6.1 $\frac{AP}{\sin(90° + \theta)} = \frac{a}{\sin\theta} \Rightarrow AP = \frac{a\cos\theta}{\sin\theta} = \frac{a}{\frac{\sin\theta}{\cos\theta}} = \frac{a}{\tan\theta}$



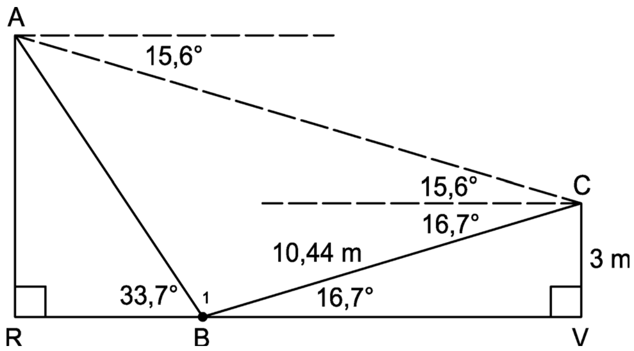
6.2 $\sin 2\theta = \frac{PR}{AP}$

$\therefore PR = AP\sin 2\theta = \frac{a\cos\theta}{\sin\theta}\cdot 2\sin\theta\cos\theta$

$= 2a\cos^2\theta$

6.3 $\frac{PQ}{PR} = \frac{a}{2a\cos^2\theta} = \frac{2}{(\cos 30)^2} = \frac{2}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\frac{3}{4}} = \frac{8}{3}$

7.



$\frac{BC}{\sin 90°} = \frac{3}{\sin 16{,}7}$ $\therefore BC = 10{,}44$

$\hat{B}_1 = 180° - 33{,}5° - 16.7°$
$= 128{,}8°$

$\hat{A}_1 = 180° - 129{,}8° - 16{,}7° - 15{,}6°$
$= 17{,}9°$

$\frac{AB}{\sin 32{,}3} = \frac{10{,}44}{\sin 17{,}9°}$ $\therefore AB = \frac{10{,}44\sin 32.3°}{\sin 17{,}9} = 18{,}2$

$\frac{AR}{\sin 33{,}5°} = \frac{18{,}2}{\sin 90°}$ $AR = \frac{18{,}2\sin 33{,}5}{1}$

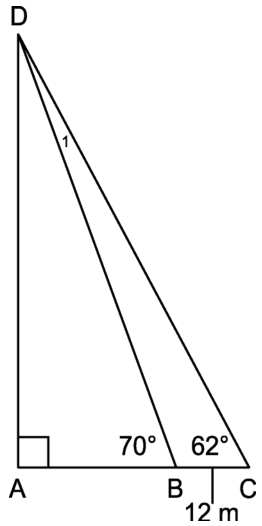
$= 10$ m (9,9 m)

8. $\hat{D}_1 = 8°$

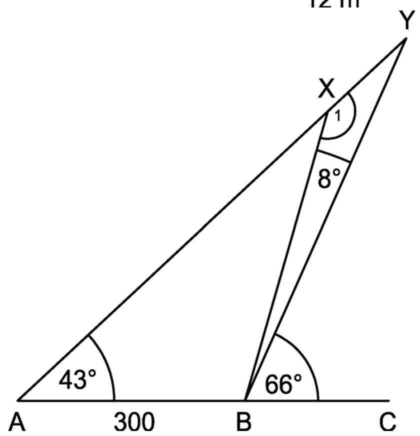
$\frac{BD}{\sin 62°} = \frac{12}{\sin 8}$

$BD = \frac{12 \sin 62}{\sin 8} = 76,1$

$\frac{AD}{\sin 70°} = \frac{76,1}{\sin 90°}$ $\therefore AD = 71,5$



9. $\hat{Y} = 23°$ $\hat{X}_1 = 149°$



$\frac{YB}{\sin 43} = \frac{800}{\sin 23}$

$\therefore YB = \frac{800 \sin 43°}{\sin 23°} = 1\ 396,3$

$XY = \frac{1\ 396,3 \sin 8°}{\sin 149°} = 377,3$

Lesson 25

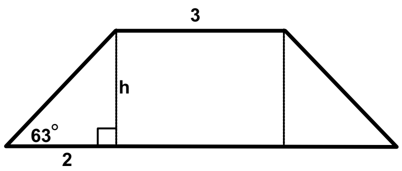
Activity 1

1. Volume:	a)	4 cm^3	b)	5 cm^3
	c)	8 cm^3	d)	7 cm^3
	e)	14 cm^3	f)	10 cm^3
Surface area;	a)	18 cm^2	b)	20 cm^2
	c)	28 cm^2	d)	24 cm^2
	e)	48 cm^2	f)	32 cm^2

2. a) $10\ m^3$ (b) $107\ cm^3$ (c) $3{,}30\ cm^3$

d) $113{,}1\ cm^3$ (e) $392{,}7\ cm^3$

f)



$\therefore h = \frac{2 \sin 63}{\sin 23}$ V = (0,50(10)(4,6) × 6

$= 4{,}6$ $= 1\ 380\ cm^3$

$\frac{h}{\sin 63} = \frac{2}{\sin 23}$

g) $\frac{1}{2}\pi(36)15$ (h) $36 = 9 + 16 - 2(3)(4) \cos \alpha$

$= 848{,}2\ cm^3$ $\cos \alpha = \frac{25 - 36}{24}$

$\alpha = 117{,}3°$

$V = \frac{1}{2}(4)(3) \sin 117{,}3 \times 3$

$= 16\ cm^3$

3. a) $197 = 80 + 10h + 16h$

$117 = 26h$

$h = 4{,}5$ cm

V = 8 x 5 x 4,5

$= 180\ cm^3$

(b) Volume P = $\pi(16)4$

$= 201{,}1\ cm^3$

Volume Q = $\pi(9)(7)$

$= 197{,}9\ cm^3$

$\therefore$ Tin P holds the most cat food.

c) Volume of water

$= \pi(9)(8)$

$= 226{,}2\ cm^3$

$226{,}2 = \pi(16)h$

$h = 4{,}5$ cm

(d) $V = \pi r^2 h$

$346 = \pi(3{,}6)2h$

$h = 8{,}5$

Surface area = $2\pi r^2 + 2\pi rh$

$= 2\pi(3{,}6)2 + 2\pi(3{,}6)(8{,}5)$

$= 273{,}7\ cm^2$

e) a) Inside area = $2\pi(15)(150)$

$= 14\ 137{,}2\ cm^2$

b) Outside curved area = $2\pi(20)(150)$

$= 18\ 849{,}6\ cm^2$

Activity 2

1. To find the weight of the pyramid, we need the volume of the pyramid.

   Volume of the pyramid = $\frac{1}{3}$ × (Base Area) × Height = $\frac{1}{3}(776)^2 \times 481$ = 96 548 885,33 ft^3

   The weight of a block of limestone is 167 pounds per cubic foot. So the weight of the pyramid is 96 548 885,33 ft^3 × 167 p/ft^3 = 1,612 366 385 × 10^{10} pounds.

   The weight in kilogram will be 1,612 366 385 × 10^{10} × 0,453 592 37 = 7 313 570 899 kg

2. a. Slant height = $\sqrt{64 + 36} = \sqrt{100} = 10$

   Surface area of pyramid

   $= \text{Area of base} + \frac{1}{2}(\text{perimeter of base} \times \text{slant height})$

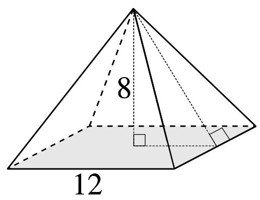
   $= 144 + \frac{1}{2}(48 \times 10)$

   $= 384\ cm^2$

   Volume $= \frac{1}{3} \times (\text{Base Area}) \times \text{Height}$

   $= \frac{1}{3} \times (144) \times 8$

   $= 384\ cm^3$



   b. Slant height = $\sqrt{15^2 - 10^2} = \sqrt{125} = 5\sqrt{5}$

   Surface area of pyramid

   $= \text{Area of base} + \frac{1}{2}(\text{perimeter of base} \times \text{slant height})$

   $= 400 + \frac{1}{2}(80 \times 5\sqrt{5})$

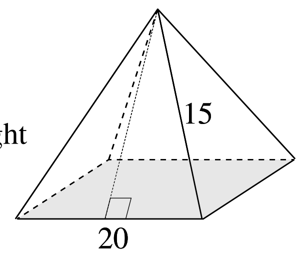
   $= 847,21\ cm^2$

   Height of pyramid $= \sqrt{(5\sqrt{5})^2 - 10^2} = 5$

   Volume $= \frac{1}{3} \times (\text{Base Area}) \times \text{Height}$

   $= \frac{1}{3} \times (400) \times 5$

   $= 666,67\ cm^3$



3. From Pythagoras: BC $= \sqrt{25 - 9} = \sqrt{16} = 4$

   So the base will have length $4\sqrt{2}$.

   Slant height $= \sqrt{(2\sqrt{2})^2 + 9} = \sqrt{17}$

   Surface area of pyramid

   $= \text{Area of base} + \frac{1}{2}(\text{perimeter of base} \times \text{slant height})$

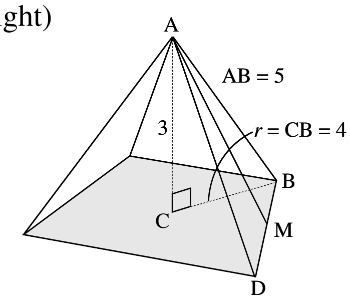
   $= (4\sqrt{2})^2 + \frac{1}{2}(4 \times 4\sqrt{2} \times \sqrt{17})$

   $= 78,65\ cm^2$

   Volume $= \frac{1}{3} \times (\text{Base Area}) \times \text{Height}$

   $= \frac{1}{3} \times (32) \times 3$

   $= 32\ cm^3$



Lesson 26

1. a) $56{,}6\ m^3$ (b) $56{,}6\ m^2$
2. a) $2\ 450{,}4\ m^2$ (b) $13\ 627{,}18\ m^3$
3. a) $5 \times 10^{-3}\ m^3$ (b) $79{,}2\ m^3$
   c) 15 714 bricks
4. a) 8,5 cm (b) $25{,}5\ cm^2$
   c) $426\ cm^2$ (d) $528\ cm^3$
5. 148,2 million
6. Volume: $1\ 466{,}08\ cm^3$
   Surface area: $837{,}07\ cm^2$
7. (a) 18,85 cm (b) 3 cm
   (c) 11,62 cm (d) $109{,}52\ cm^3$

Lesson 27

Activity 5

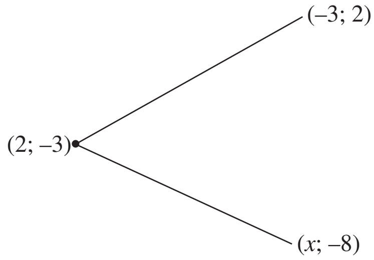
1. a) $\sqrt{(0-2)^2 + (y-0)^2} = \sqrt{40}$

$\therefore 4 + y^2 = 40$

$\therefore y^2 = 36$

$\therefore y = \pm 6$

1. (b)



$(2+3)^2 + (-3-2)^2 = (x-2)^2 + (-8+3)^2$

$5^2 + 5^2 = (x-2)^2 + 5^2$

$\therefore (x-2)^2 = 25$

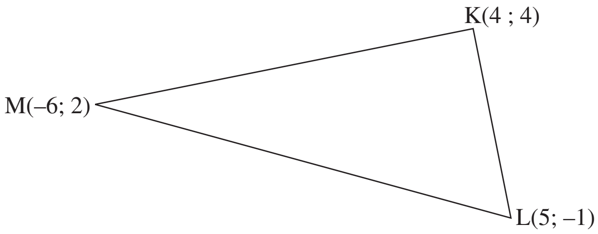
$x - 2 = \pm 5$

$\therefore x = 2 \pm 5$

$\therefore x = 7$ or $x = -3$

1. c) (3; 2) d) $y = -6$ or $y = 2$ 2. D(19;–11)

3.



$M_{KL} = \frac{y_K - y_L}{x_K - x_L} = \frac{4 - (-1)}{4 - 5} = \frac{5}{-1} = -5$

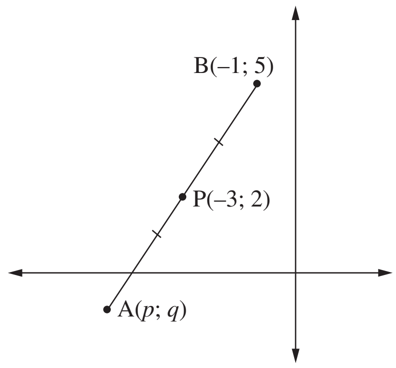
$M_{KM} = \frac{y_K - y_M}{x_K - x_M} = \frac{4 - 2}{4 - (-6)} = \frac{2}{10} = \frac{1}{5}$

Thus $M_{KL} \cdot M_{KM} = -5\left(\frac{1}{5}\right) = -1$

$\therefore$ KL⊥KM

Therefore ΔKLM is right angled at K.

4.



P is the midpoint of AB

$x_P = \frac{x_A + x_B}{2}$ and $y_P = \frac{y_A + y_B}{2}$

$\therefore -3 = \frac{p + (-1)}{2}$ and $2 = \frac{q + 5}{2}$

$\therefore -6 = p - 1$ and $4 = q + 5$

$\therefore p = -5$ and $q = -1$

Thus A(p; q) = A(–5; –1)

5. $M_{DE} = \frac{y_D - y_E}{x_D - x_E} = \frac{3 - (-1)}{2 - (-3)} = \frac{4}{5}$ and $M_{DF} = \frac{y_D - y_F}{x_D - x_F} = \frac{3 - (-2)}{2 - 6} = \frac{5}{-4}$

$\therefore (M_{DE}) \times (M_{DF}) = \left(\frac{4}{5}\right)\left(-\frac{5}{4}\right) = -1$

$\therefore$ DE⊥DF

$\therefore \hat{D} = 90°$.

Furthermore: P is the midpoint of DE and Q the midpoint of DF

Thus M_{dpt} DE: $x_P = \frac{1}{2}(x_D + x_E)$ and $y_P = \frac{1}{2}(y_D + y_E)$

$\therefore x_P = \frac{1}{2}(2 + (-3))$ $\quad y_P = \frac{1}{2}(3 + (-1))$

$\therefore x_P = \frac{1}{2}(-1)$ $\quad y_P = \frac{1}{2}(2)$

$\therefore x_P = -\frac{1}{2}$ $\quad y_P = 1$

$\therefore P\left(-\frac{1}{2}; 1\right)$

M_{dpt} DF: $x_Q = \frac{1}{2}(x_D + x_F)$ and $y_Q = \frac{1}{2}(y_D + y_F)$

$\therefore x_Q = \frac{1}{2}(2 + 6)$ and $y_Q = \frac{1}{2}(3 + (-2))$

$\therefore x_Q = \frac{1}{2}(8)$ and $y_Q = \frac{1}{2}(1)$

$\therefore x_Q = 4$ and $y_Q = \frac{1}{2}$

$\therefore Q\left(4; \frac{1}{2}\right)$

Now: PQ will be parallel to DE if and only if $M_{PQ} = M_{DE}$

$M_{PQ} = \frac{y_P - y_Q}{x_P - x_Q} = \frac{1 - \frac{1}{2}}{-\frac{1}{2} - 4} \cdot \frac{2}{2} = \frac{2 - 1}{-1 - 8} = -\frac{1}{9}$ and $M_{EF} = \frac{y_E - y_F}{x_E - x_F} = \frac{-1 - (-2)}{-9} = -\frac{1}{9}$

$\therefore M_{PQ} = M_{EF}$

$\therefore$ PQ||EF

Now for the lengths:

$PQ = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$ $\quad EF = \sqrt{(x_E - x_F)^2 + (y_E - y_F)^2}$

$= \sqrt{\left(-\frac{1}{2} - 4\right)^2 + \left(1 - \frac{1}{2}\right)^2}$ $\quad = \sqrt{(-3 - 6)^2 + (-1 - (-2))^2}$

$= \sqrt{\left(-\frac{9}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$ $\quad = \sqrt{(81)^2 + (1)^2}$

$= \sqrt{\frac{82}{4}}$ $\quad = \sqrt{82}$

$= \frac{\sqrt{82}}{2}$

thus 2PQ = EF

6. $y = 2x + 1 \quad y = 3x + 6 \quad 2y = -6x + 7$

$y = -3x + \frac{7}{2}$

and $y = 3x + 1$

Two lines have gradients that are equal to 3, so these will be parallel.

For ① = ②: $2x + 1 = 3x + 6$ $\quad -3x + \frac{7}{2} = 3x + 6$

$x = -5$ $\quad -6x + 7 = 6x + 12$

Then $y = -10 + 1$ $\quad -12x = 5$

$= -9$ $\quad x = -\frac{5}{12}$

So A(–5; –9) $\quad$ Then $y = 3\left(\frac{-5}{12}\right) + 6 = \frac{19}{4}$

$B\left(-\frac{5}{12}; \frac{19}{4}\right)$

$3x + 1 = 2x + 1$ $\quad 3x + 1 = -3x + \frac{7}{2}$

$x = 0$ $\quad 6x + 2 = -6x + 7$

Then $y = 1$ $\quad 12x = 5$

$c(0; 1)$ $\quad x = \frac{5}{12}$

$\therefore D\left(\frac{5}{12}; \frac{9}{4}\right)$

Then $y = 3\left(\frac{5}{12}\right) + 1$

$= \frac{5}{4} + \frac{4}{4}$

$= \frac{9}{4}$

Lesson 28

1. a) $M_{AB} = \frac{8 - 0}{6 + 3} = \frac{8}{9}$

$\tan\theta = \frac{8}{9}$

$\theta = 41{,}6°$

b) $M_{AB} = \tan\theta$

$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{2} - 1}{-4 - 7}\right)$

$= \tan^{-1}(-0{,}0376)$

$= 177{,}8°$

2. a) $m = -\frac{1}{3} \tan\theta = -\frac{1}{3}$

$\theta = 161{,}6°$

b) $\tan\theta = \frac{1}{2}$ $\therefore \theta = 23{,}6°$

c) $\tan\theta = -4$ $\therefore \theta = 104°$

d) $\tan\theta = 2$ $\therefore \theta = 63{,}4°$

3. $M_{DE} = \frac{-2-7}{5+3} = -\frac{9}{8}$

$M_{EF} = \frac{5-7}{2+3} = \frac{-2}{5}$

$M_{DF} = \frac{5+2}{2-5} = \frac{7}{-3}$

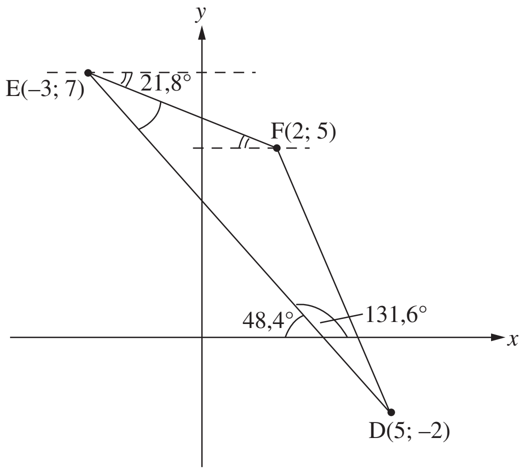
$\theta_{DE} = \tan^{-1}\left(-\frac{9}{8}\right)$

$= 131{,}6°$

$\theta_{EF} = \tan^{-1}\left(-\frac{2}{5}\right)$

$= 158{,}2°$

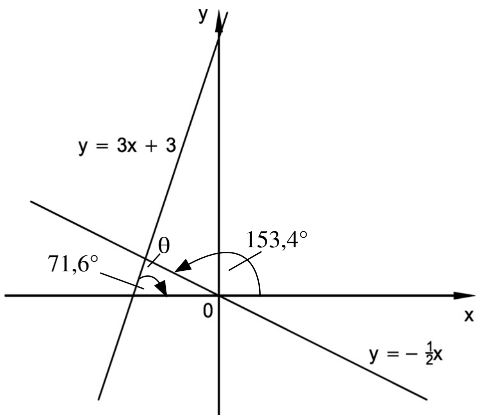
$\therefore$ DEF $= 48{,}4° - 21{,}8° = 26{,}6°$



4. Draw the graph

$\alpha = 71{,}6°$

$\beta = -26{,}6° + 180° = 153{,}4°$



Then: $\theta = 153{,}4° - 71{,}6°$ (ext ∟ of △)

$= 81{,}8°$

OR $\theta' = 180° - 81{,}8° = 98{,}2°$

5. $\tan\theta = \frac{1}{4}$ $\theta = 14°$

$\tan\alpha = -2$ $\alpha = 63{,}4°$

$\therefore \alpha = 102{,}6°$ $\beta = 77{,}4°$

6. $\tan\alpha = 1$ $\alpha = 45°$

$\therefore \beta = 90°$ and the gradient is undefined.

7. $\tan\beta = -2 \therefore \beta = 116{,}6°$

$\therefore \alpha = 54{,}6°$

m = 1,4

8. $A\hat{O}B = 53{,}1°$ $C\hat{O}B = 26{,}6°$

$\tan 26{,}6° = \frac{1}{2}$ gradient of OC $= \frac{1}{2}$

9. $\hat{P} = 101{,}3°$ $\hat{Q} = 54{,}3°$ $\hat{R} = 24{,}4°$

Lesson 29

1. a) $m_{CD} = m_{AB}$

$\frac{t + 1 - 3}{-\frac{1}{3} - 2} = \frac{3}{2}$

$2(t - 2) = 3(-\frac{1}{3} - 2)$

$2t - 4 = -1 - 6$

$2t = -3$

$t = -\frac{3}{2}$

b) $m_{AB} = m_{AE}$

$\frac{3}{2} = \frac{3}{r - 3}$

$3(r - 3) = 6$

$r - 3 = 2$

$r = 5$

2. $(3 - 2k)x + (k + 1y) = 12$

or $(k + 1)y = -(3 - 2k)x + 12$

$y = \frac{(-3 + 2k)x}{k + 1} = \frac{12}{k + 1}$

a) $\frac{-3 + 2k}{k + 1} = 4$

$-3 + 2k = 4k + 4$

$-7 = 2k$

$k = -\frac{7}{2}$

b) $m = \frac{1}{-3}$

$m_{perp} = 3$

$\therefore \frac{-3 + 2k}{k + 1} = 3$

$-3 + 2k = 3k + 3$

$-6 = k$

c) Substitute $x = -3$ $y = 4$

$(3 - 2k)(-3) = (k + 1)\ 4 = 12$

$-9 + 6k + 4k + 4 = 12$

$10k = 17$

$k = \frac{17}{10}$

d) $3 - 2k = 0$

$2k = 3$

$k = \frac{3}{2}$

e) $k + 1 = 0$

$k = -1$

3. a) $y = 7$ or $y = -5$

b) $x = 3$ or $x = -5$

Lesson 30

1. $m_{MN} = 1$

$\tan a = 1$ $\alpha = 45°$

$m_{PQ} = -\frac{3}{2}$ $\beta = 180° - 56{,}3°$

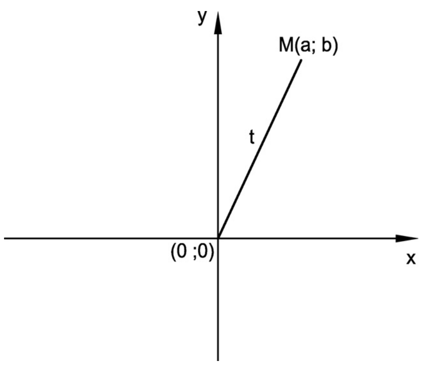
$= 123{,}7°$

$\theta + \alpha = \beta$

$\theta = \beta - \alpha$

$= 78{,}7°$

2. $a^2 + b^2 = t^2$



Point is on the line, so $a + \sqrt{3}\ b = 2t$

$b = \frac{2t - a}{\sqrt{3}}$ Substitute

$a^2 + \left(\frac{2t - a}{\sqrt{3}}\right)^2 = t^2$

$a^2 + \frac{4t^2 - 4at + a^2}{3} = t^2$

$3a^2 + 4t^2 - 4at + a^2 = 3t^2$

$4a^2 - 4at + t^2 = 0$

$(2a - t)(2a - t) = 0$

$a = \frac{t}{2} \quad b = \frac{3t}{2\sqrt{3}}$

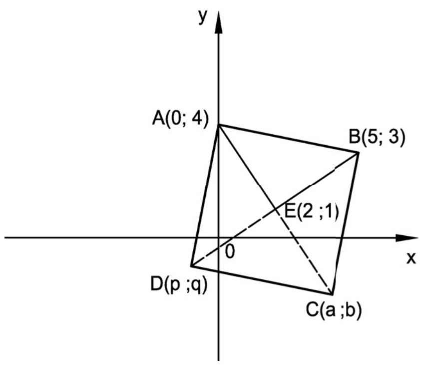
3. a) $m_{AE} + \frac{3}{-2} \qquad m_{BE} = \frac{2}{3} \qquad \therefore \hat{E} = 90°$

b) $2 = \frac{a + 0}{2} \qquad 1 = \frac{b + 4}{2}$

$\therefore a = 4 \qquad b = -2 \qquad C(4\ ;\ -2)$

$\frac{p + 5}{2} = 2 \qquad \frac{q + 3}{2} = 1$

$p = -1$ q $= -1$ D$(-1\ ;\ -1)$



4. a) T(1 ; 1)

b) T(1 ; 1) gradient PR $= -1$

gradient perp. bisector $= 1$

$y - 1 = x - 1 \quad y = x$

If S is on the line $a = b$

c) $\frac{1}{2}$ PR × TS = 12

$\frac{1}{2}\sqrt{32}.\sqrt{2(1 - a)^2} = 12$

$32.2(1 - a)^2 = 24^2$

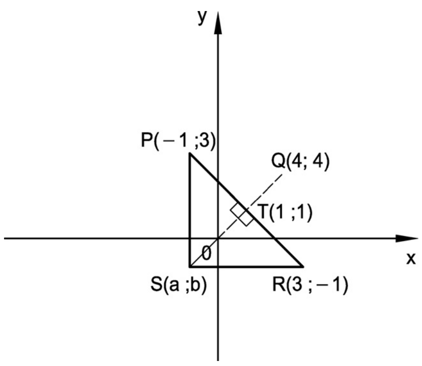
$(1 - a)^2 = 9$

$a^2 - 2a - 8 = 0$

$(a - 4)(a + 2) = 0$

$a = 4$ or $a = -2$

S(–2 ; –2)



d) Prove diagonals bisect at 90°

i) Mid-point QS is (1 ; 1)

Mid-point PR is T(1 ; 1)

ii) PR ⊥ QS

$\therefore$ PQRS is a rhombus

5. a) $a = \frac{-2+8}{2}$ $a = 3$

b) $d^2 = 25$

$e^2 + 16 = 25$

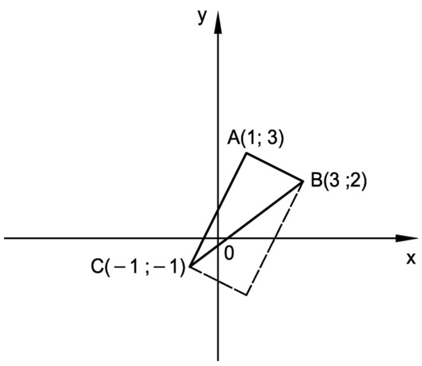
$e^2 = 9 \therefore e = \pm 3$

c) i) pt$(-1 ; -1)$ $m_{AB} = -\frac{1}{2}$

Equation: $y + 1 = -\frac{1}{2}(x + 1)$

$2y + 2 = -x - 1$

$2y = -x - 3$



ii) Pt$(3 ; 2)$ $m_{AB} = -\frac{1}{2}$

$m_{perp} = 2$

Equation : $y - 2 = 2(x - 3)$

$y = 2x - 4$

iii) $y = 2x - 4$

$2(2x - 4) = -x - 3$

$4x - 8 = -x - 3$

$5x = 5$

$x = 1$ $y = -2$

$(1 ; -2)$

6. a) $m_{AD} = m_{BC}$

$\frac{4}{-2} = \frac{6-y}{x-4}$

$-2(x - 4) = 6 - y$

$-2x + 8 = 6 - y$

$y = 2x - 2$

$BC^2 = 4AD^2$

$(x - 4)^2 + (6 - y)^2 = 4[4 + 16\}$

$(x - 4)^2 + (6 - y)^2 = 80$

Solve simultaneously.

$(x - 4)^2 + (6 - 2x + 2)^2 = 80$

$x^2 - 8x + 16 + 64 - 32x + 4x^2 = 80$

$5x^2 - 40x = 0$

$5x(x - 8) = 0$

$x = 0$ or $x = 8$ $(x < 4)$

$\therefore x = 0$ $y = -2$

b) $A(-4 ; 3)$ $B(0 ; 6)$

$m = \frac{3}{4}$ eq. $y = \frac{3}{4}x + 6$

$C(4 ; -2)$ $D(-2 ; -1)$

$m = -\frac{1}{6}$ equation $y + 1 = -\frac{1}{6}(x + 2)$

$y + 1 = -\frac{1}{6}x - \frac{1}{3}$

Solve simultaneously.

$\frac{3}{4}x + 6 + 1 = -\frac{1}{6}x - \frac{1}{3}$

$9x + 84 = -2x - 4$

$11x = -88$

$x = -8$ $y = 0$ $(-8 ; 0)$

7. a) 18,4°

b) Gradient BK is $\frac{8}{4} = 2$

$\therefore B\hat{K}X = 63,4°$

$\therefore T\hat{A}C = B\hat{A}C = 45°$ By geometry

c) Equation BAK is $y = 2x + 4$

$\therefore$ K(–2 ; 0) eq. KH is gradient $\frac{1}{\sqrt{3}}$

Equation : $y = \frac{1}{\sqrt{3}}(x + 2)$

$\therefore y = \frac{1}{\sqrt{3}}(8) = \frac{8}{\sqrt{3}}$

Lesson 31

1. Let AP = $5k$

$\therefore$ PB = $4k$

PQ $\parallel$ BC

$\therefore \frac{AP}{PB} = \frac{AQ}{QC} = \frac{5\text{ m}}{4\text{ m}}$

AQ = 5 m QC = 4 m

$\therefore$ 24 = 4 m

m = 6

$\therefore$ AQ = 30 cm

2. PR $\parallel$ BC

$\therefore \frac{AP}{PB} = \frac{AR}{RC}$ Line $\parallel$ one side of $\triangle$

RQ $\parallel$ AB

$\therefore \frac{AR}{RC} = \frac{BQ}{QC}$ Line $\parallel$ one side of $\triangle$

$\therefore \frac{AP}{PB} = \frac{BQ}{QC}$

3. i) $\frac{BS}{SE} = \frac{2}{1}$ S is the centroid

ii) In $\triangle$BEF SF $\parallel$ EF

$\therefore \frac{BS}{SE} = \frac{BD}{DF} = \frac{2k}{k}$ Line $\parallel$ one side of $\triangle$

BD = $2k$ and DF = k

But BD = DC AD is a median

$\therefore$ DC = $2k$ $\therefore$ FC = k

$\therefore \frac{BF}{FC} = \frac{3k}{k} = \frac{3}{1}$

4. In $\triangle$PQM

NK $\parallel$ PM

$\therefore \frac{PN}{NQ} = \frac{MK}{KQ} = \frac{3\text{ m}}{5\text{ m}}$ Line $\parallel$ one side of $\triangle$

let MK = 3 m and KQ = 5 m

$\therefore$ MQ = 8 m

But MQ = SM = 8 m Diagonals of a parallelogram

i) $\therefore \frac{KQ}{SQ} = \frac{5\text{ m}}{16\text{ m}} = \frac{5}{16}$

ii) $\therefore \frac{SM}{MK} = \frac{8\text{ m}}{3\text{ m}} = \frac{8}{3}$

iii) In $\triangle$NSK TM $\parallel$ NK

$\frac{SM}{MK} = \frac{ST}{TN} = \frac{8t}{3t}$

ST = $8t$

8t = 32 $\therefore t = 4$

$\therefore$ SN = $11t$

$\therefore$ SN = 44 cm

5. a) In $\triangle$ABC XY $\parallel$ BC

$\therefore \frac{AX}{XB} = \frac{AY}{YC}$ Line $\parallel$ one side of $\triangle$

In $\triangle$ACP: XZ $\parallel$ CP

$\therefore \frac{AY}{YC} = \frac{AZ}{ZP}$ Line $\parallel$ one side of $\triangle$

$\therefore \frac{AX}{XB} = \frac{AZ}{ZP}$

$\therefore$ ZP $\parallel$ BP Sides in proportion

b) In $\triangle$AMP KZ $\parallel$ MP

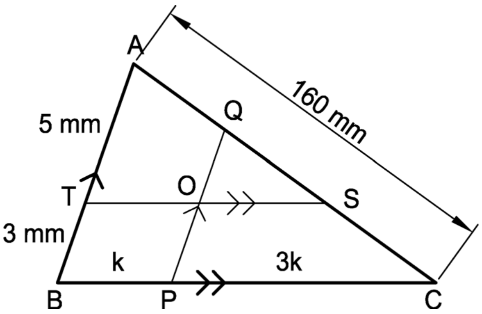
$\therefore \frac{AK}{KM} = \frac{AZ}{ZP}$ Line $\parallel$ one side of $\triangle$

In $\triangle$ACP YZ $\parallel$ CP

$\therefore \frac{AZ}{ZP} = \frac{AY}{YC}$ Line $\parallel$ one side of $\triangle$

$\therefore \frac{AK}{KM} = \frac{AY}{YC}$

9. a) TS ∥ BC



$\therefore \frac{AS}{SC} = \frac{AT}{TB} = \frac{5t}{3t}$

$\therefore$ AC = 8t

160 = 8t

t = 20

$\therefore$ AS = 100

SC = 60

AB ∥ QP

$\therefore \frac{AQ}{AC} = \frac{BP}{BC} = \frac{1}{4} = \frac{m}{4m}$

4m = 160 $\therefore$ m = 40

$\therefore$ AQ = 40 mm and QS = 60 mm

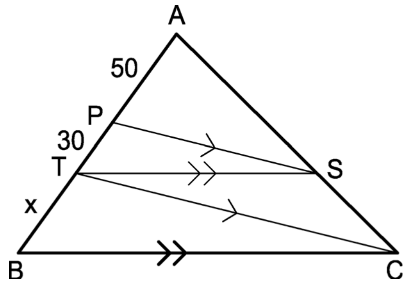
b) $\therefore$ QS = SC = 60

In △QPC: QS = SC

OS ∥ PC

$\therefore$ QO = OP

10. PS ∥ TC



$\frac{AP}{PT} = \frac{AS}{SC} = \frac{5}{3}$

TS ∥ BC

$\frac{AT}{TB} = \frac{AS}{SC} = \frac{80}{x}$

$\therefore \frac{5}{3} = \frac{80}{x}$

$\therefore 5x = 240$

$x = 48$

Lesson 32

1. a) In △ABD and △AEC

$\hat{A}_1 = \hat{A}_2$ (given)

$\hat{B} = \hat{E}$ (given)

3rd angle = 3rd angle (angles in triangle)

$\therefore$ △ABD ||| △AEC

b) $\frac{AB}{AE} = \frac{AD}{AC}$

AB.AC = AD.AE

= AD(AD + DE)

= AD^2 + AD.DE

Now prove AD.DE = BD.DC

In △s ADB & DEC

$\hat{D}_1 = \hat{D}_2$ Vertically opposite angles

$\hat{B} = \hat{E}$ given

3rd angle = 3rd angle Angles in triangle

$\therefore$ △DBA ||| △DEC

$\therefore \frac{DB}{DE} = \frac{DA}{DC}$

$\therefore$ DB.DC = DA.DE

$\therefore$ AB.AC = AD2 + BD.DC

2. a) In $\triangle$s GDF and GED

$\frac{GD}{GE} = \frac{6}{9} = \frac{2}{3}$

$\frac{DF}{DE} = \frac{3}{4{,}5} = \frac{2}{3}$

$\frac{GF}{GD} = \frac{4}{6} = \frac{2}{3}$ the sides are in proportion

$\therefore \triangle GDF \,|||\, \triangle GED$

b) $\hat{G} = \hat{G}$

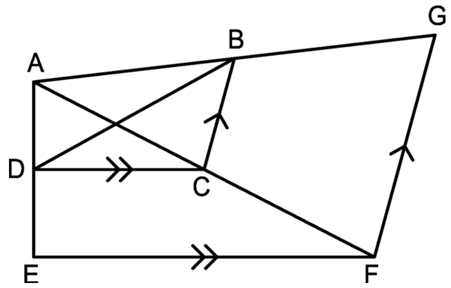
$F\hat{D}G = \hat{E}$

$D\hat{F}G = E\hat{D}G$

3. a) In $\triangle$AEF: $\frac{AC}{CF} = \frac{AD}{DE}$ (DC $\|$ EF)

$\frac{AC}{5} = \frac{12}{3}$

AC = 20cm



b) $\triangle ADV \,|||\, \triangle AEF$

$\therefore \frac{AD}{AE} = \frac{DC}{EF}$

$\therefore \frac{12}{15} = \frac{14}{EF}$

EF = 17,5 cm

c) AB = 14cm

In $\triangle$AGF: $\frac{BG}{AB} = \frac{CF}{AC}$ (BC $\|$ GF)

$\frac{BG}{14} = \frac{5}{20}$

BG = 3,5 cm

4. a) By Pythagoras QR = 52 cm

b) In $\triangle$s PQR and TSR

$\hat{R} = \hat{R}$ Common

$\hat{Q} = \hat{S}$ (90°)

$\therefore \hat{P} = \hat{T}$ Angles in triangle

$\therefore \triangle RPQ \,|||\, \triangle RST$

$\therefore \frac{RQ}{RS} = \frac{QP}{ST} = \frac{RP}{RT}$

$\therefore \frac{32}{20} = \frac{24}{ST} = \frac{40}{RT}$

$ST = \frac{24 \times 20}{32}$

= 15 cm

c) $RT = \frac{40 \times 20}{32} = 25$

$\therefore$ QT = 7 cm

5. In $\triangle$s ABD and BCD

$\hat{D} = \hat{D}$ (common)

$\hat{A} = \hat{B}$ (given)

$\therefore \hat{B} = \hat{C}$ Angles in triangle

$\therefore \triangle DAB \,|||\, DBC$

$\therefore \frac{AD}{DB} = \frac{AB}{BC} = \frac{BD}{CD}$

$\frac{P + 6}{15} = \frac{15}{6}$

$P = \frac{15 \times 15}{6} - 6$

P = 31,5

$\frac{AB}{5} = \frac{15}{6}$

$AB = 12\frac{1}{2}$

6. a) In $\triangle$s BCA and BCN

$\hat{C} = \hat{C}$ (common)

$\hat{B} = \hat{N}$ (given)

$\therefore \hat{A} = \hat{B}$ ($\angle$s in $\triangle$)

$\therefore \triangle$ CBA $|||$ $\triangle$ CNB

$\therefore \frac{CB}{CN} = \frac{AC}{CB}$

$\therefore BC^2 = AC.CN$

b) In $\triangle$s AMN and NCB

$\hat{M}^1 = \hat{N}$	($\hat{M}1 = \hat{B}$	Corresponding angles MN $\|\|$ BC
$\hat{N}^1 = \hat{C}$		Corresponding angles MN $\|\|$ BC
$\therefore \hat{A} = \hat{B}$		Angles in triangle

$\therefore \triangle$ MNA $||$ $\triangle$ NCB

$\therefore \frac{MN}{NC} = \frac{AM}{NB}$

$\therefore$ MN.NB = AM.NC

7. In $\triangle$s BMN and BAC

$\hat{B} = \hat{B}$ (common)

$\hat{M} = \hat{A}$	Corresponding angles AC $\|\|$ MN
$\hat{N} = \hat{C}$	Corresponding angles AC $\|\|$ MN

$\therefore \triangle BMN ||| \triangle BAC$

$\frac{BM}{BA} = \frac{MN}{AC} = \frac{BN}{BC}$

$\frac{6}{18} = \frac{7,5}{AC}$

AC = 22,5

$\frac{x}{x + 9} = \frac{1}{3}$

$2x = x + 9$

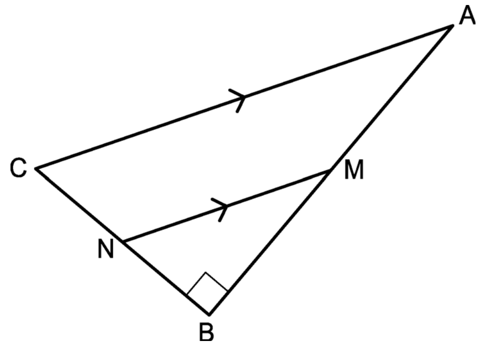
$x = 4,5$

$AC^2 = AB^2 + BC^2$

$\therefore$ The $\triangle$ is right angled

(Interesting – the diagram is not an accurate sketch but it is only a sketch and this often happens in mathematics.)

Possibly should look like this.



8. $\frac{BC}{AC} = \frac{CD}{AD}$

$\frac{BC}{AC} = \frac{BD}{CD}$

$\therefore \frac{BC}{AC} = \frac{CD}{AD} = \frac{BD}{CD}$

$\triangle$BCD is similar to $\triangle$ACD

$\therefore \hat{D}^2 = \hat{D}^1$

$\hat{B} = \hat{C}_1 = x$

$\hat{C}_2 = \hat{A} = y$

$\therefore 2x + 2y = 180°$

$x + y = 90°$

$\therefore \hat{C} = 90°$

By Pythagoras

$AB^2 = AC^2 + BC^2$

Lesson 33

1. $PQ^2 = QS.QR$ (perp. from right < of $\triangle$ to hyp.)

$PR^2 = RS.RQ$ (perp. from right < of $\triangle$ to hyp.)

$\therefore \frac{PQ^2}{PR^2} = \frac{QS.QR}{RS.RQ} = \frac{QS}{RS}$

2. i) RT × RP (perp. from right < to hyp.)

ii) $QT^2 + TR^2$ (Pythagoras)

iii) $PR^2 - PQ^2$ (Pythagoras)

iv) PT × PR (perp. from right < to hyp.)

v) $QT^2 + PT^2$ (Pythagoras)

vi) $PR^2 - QR^2$ (Pythagoras)

vii) TR × TP (perp. from right < to hyp.)

3. In $\triangle$DAC: $DB^2 = BA.BC$ (perp. from right-angle to hypotenuse)

$\therefore 4 = BA.4$

$\therefore BA = 1$

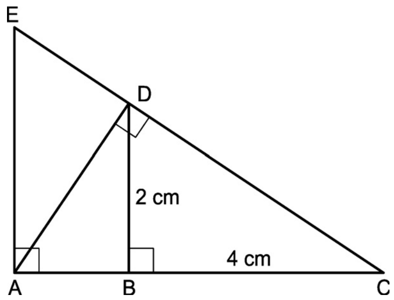
BD ∥ AE ((B)1 = (A) (angles)

$\therefore \triangle CDB\ ///\ \triangle CEA$

$\therefore \frac{DB}{AE} = \frac{CB}{AC}$

$\therefore \frac{2}{AE} = \frac{4}{5}$

$\therefore AE = \frac{10}{4} = \frac{5}{2} = 2\frac{1}{2}$ units



4. DK = x KC = $3x$

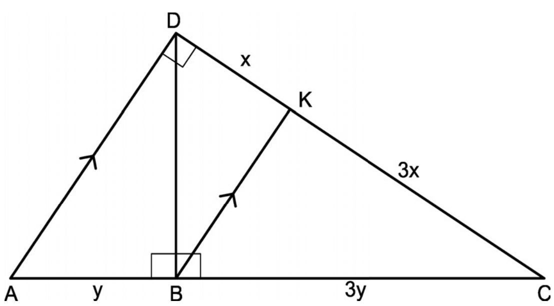
$\frac{DK}{KC} = \frac{AB}{BC} = \frac{1}{3}$

AB = y BC = $3y$

$DA^2 = AB.AC$ (perp. from right < to hyp.)

$DA^2 = y.4y$

$DA2 = 4y^2$

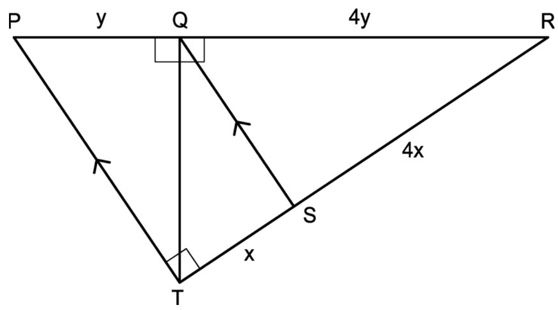


$DA = 2y$

$2DA = 4y$

5. $\frac{TS}{SR} = \frac{PQ}{QR} = \frac{1}{4}$ (PT ∥ SQ)

$PQ = y$ $QR = 4y$



In △ TPR

$TP^2 = PQ.PR$ (perp. from right < to hyp.)

$TP^2 = y.5y$

$= 5y^2$

$TP = \sqrt{5y}$

$4TP = \sqrt{5}\ 4y$

$4TP = \sqrt{5}\ QR$

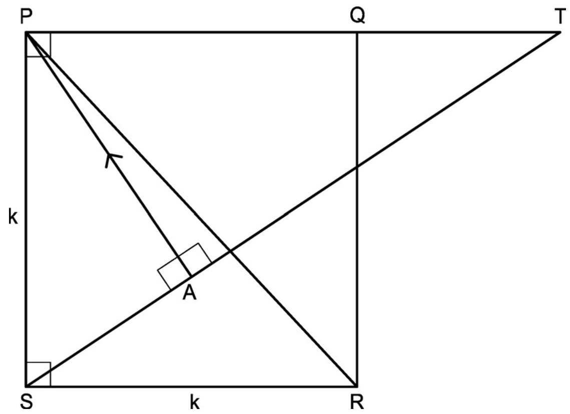
6. let $PS = k$ $\therefore$ $SR = k$ (square)

i) $\therefore$ $PR^2 = k^2 + k^2$ (Pythagoras)

$\therefore$ $PR = R^2 K$

$\therefore$ $PR = R^2 PS$

ii) In △ PTS: $PT^2 = TA \cdot TS$ (perp. from right < to hyp.)



$\therefore$ $PR^2 = TA \cdot TS$ (PT = PR)

$\therefore$ $2PS^2 = TA \cdot TS$ (PR = R^2 PS from (i)

$\therefore$ $PS^2 = \frac{TA \cdot TS}{2}$

but $PS^2 = SA.ST$ (perp. from right < to hyp.)

$\therefore$ $AS \cdot TS = \frac{AT \cdot TS}{2}$

$\therefore$ $2AS \cdot TS = AT \cdot TS$

$\therefore$ $2AS = TA$

7. i) $BC^2 = BD \cdot BA$ (perp. from right < to hyp.)

$AC^2 = AD \cdot AB$ (perp. from right < to hyp.)

$\therefore$ $\frac{BC^2}{AC^2} = \frac{BD \cdot BA}{AD \cdot BA}$

$= \frac{BD}{AD}$

ii) $\frac{BC^4}{AC^4} = \frac{BD^2}{AD^2}$ from above

$BD^2 = BE \cdot BC$ (perp. from right < to hyp.)

$AD^2 = AF \cdot AC$ (perp. from right < to hyp.)

$\therefore \frac{BC^4}{AC^4} = \frac{BE \cdot BC}{AF \cdot AC}$

$\therefore \frac{BC^3}{AC^3} = \frac{BE}{AF}$

Lesson 34

Activity 1

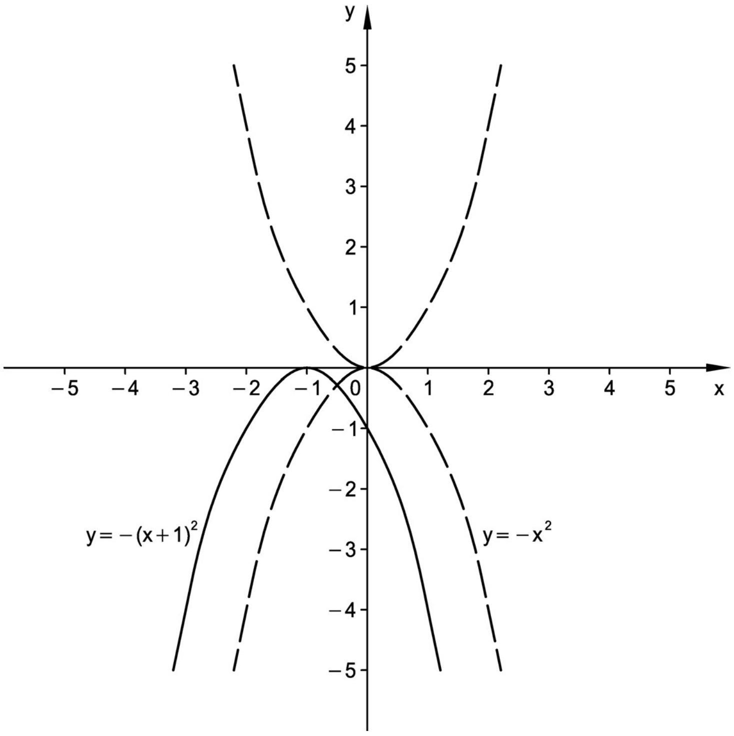
1. a) $y = 2(x + 3)^2$

   $y = 2x^2$ is translated 3 units to the left horizontally.

   b) $y = -3(x - 4)^2$

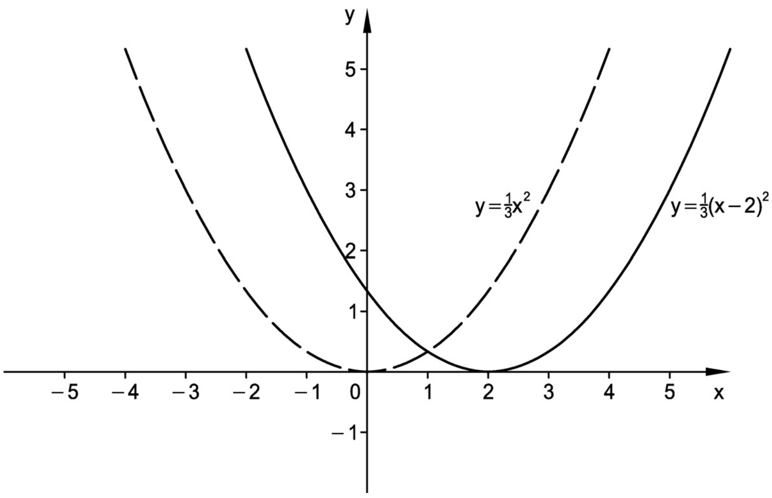
   $y = 3x^2$ is reflected across the x-axis and translated 4 units to the right horizontally.

2. a)

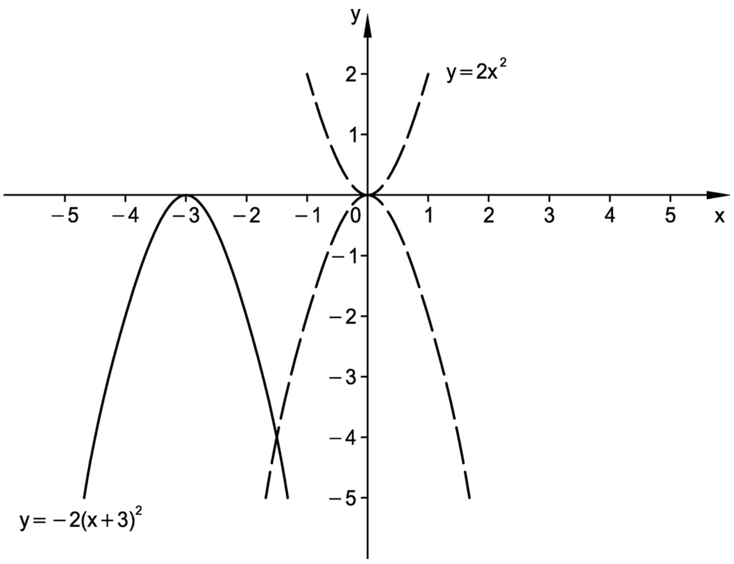


$y = x^2$ is reflected across the x–axis and translated 1 unit to the left horizontally.

b)



c)



$y = 2x^2$ is reflected across the x–axis and translated 3 units to the left.

Activity 2

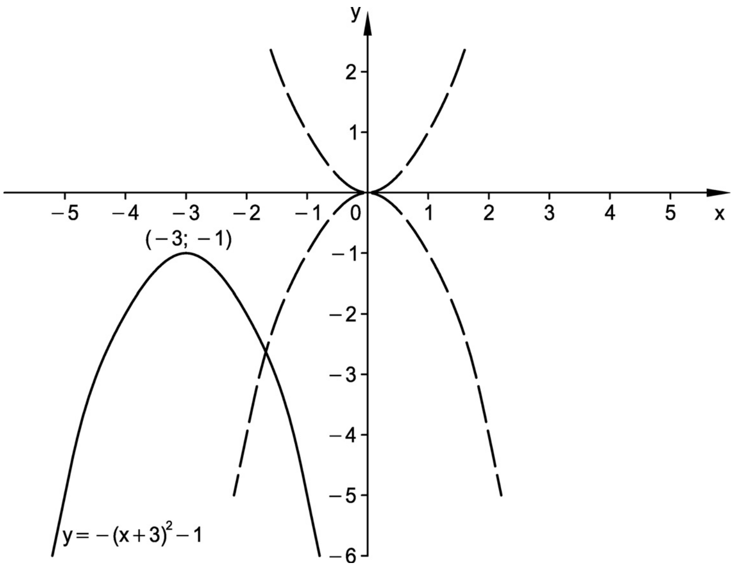
1. a) $y = 2(x - 2)^2 - 1$

$y = 2x^2$ is translated 1 unit down and 2 units to the right horizontally.

b) $y = -\frac{1}{3}(x - 3)^2 - 1$

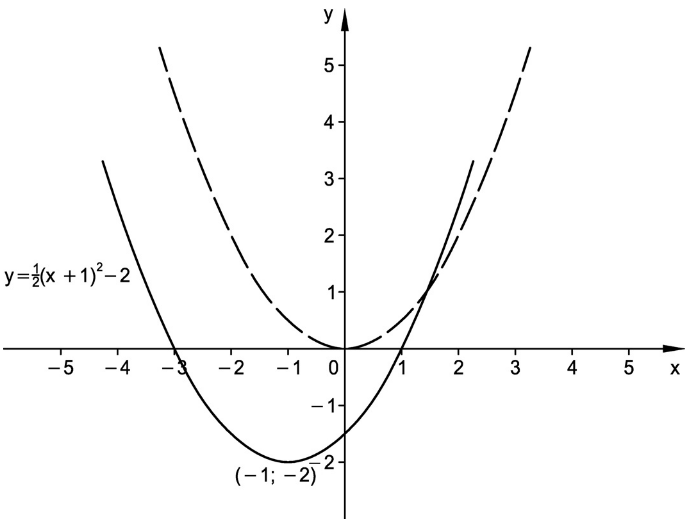
$y = ax^2$ is reflected across the x–axis, translated 1 unit down and 3 units to the right.

2. a)



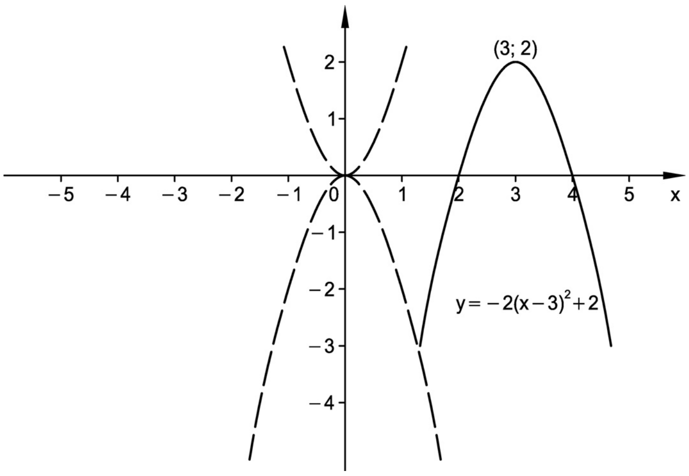
$y = x^2$ is reflected across the x–axis, translated 3 units to the left and 1 unit down.

b)



$y = \frac{1}{2}x^2$ is translated 1 unit to the left and two units down.

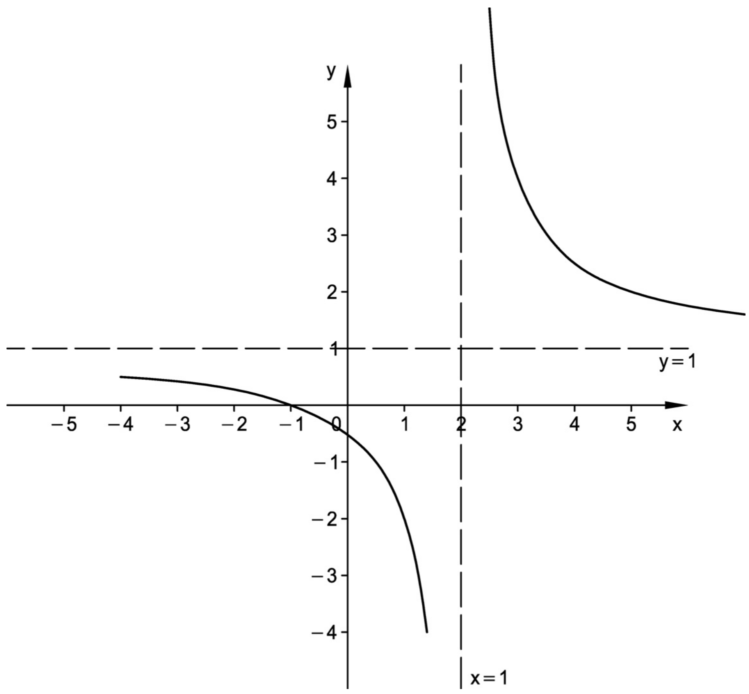
c)



$y = 2x^2$ is reflected across the x–axis, translated 3 units to the right and up 2 units.

Activity 3

1. a)

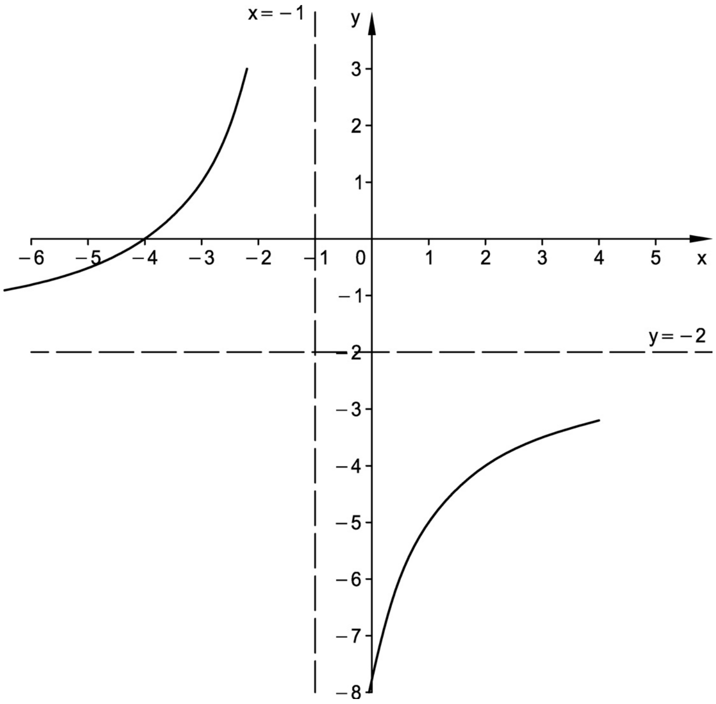


x int. $(-1 ; 0)$

y int. $\left(0 ; -\frac{1}{2}\right)$

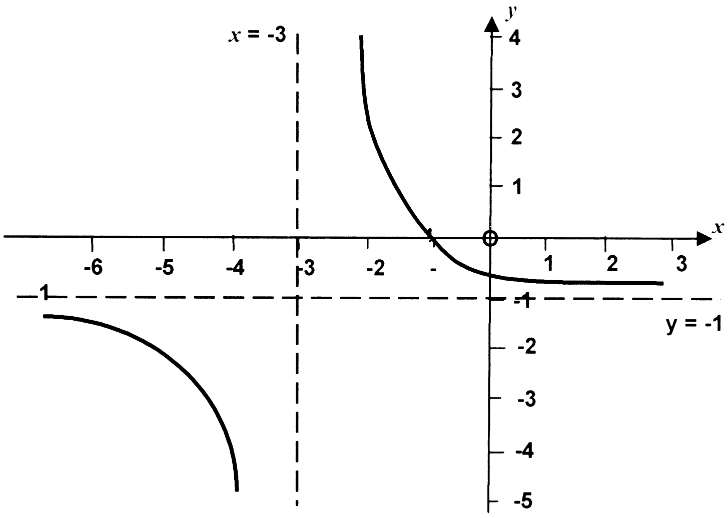
$y = \frac{3}{x}$ is translated 2 units to the right and 1 unit up.

b)



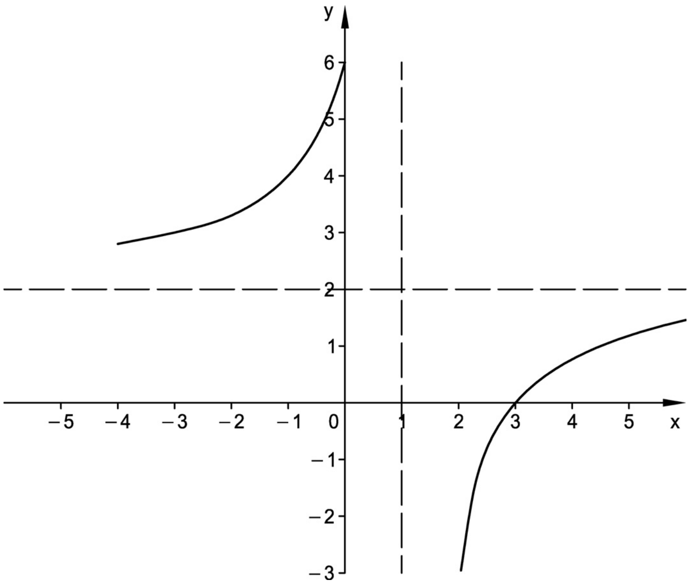
$y = \frac{6}{x}$ is reflected across the y–axis, translated 1 unit to the left and 2 units down.

c)



y int. $= \left(0\ ;-\frac{1}{3}\right)$

d)

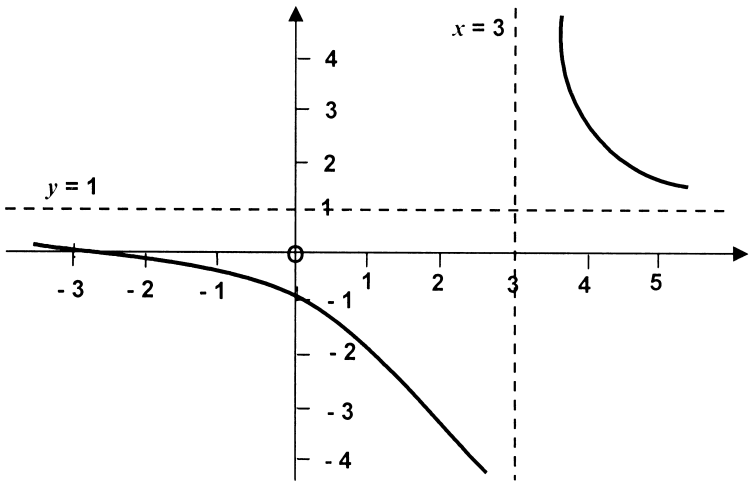


x int. (3 ; 0)

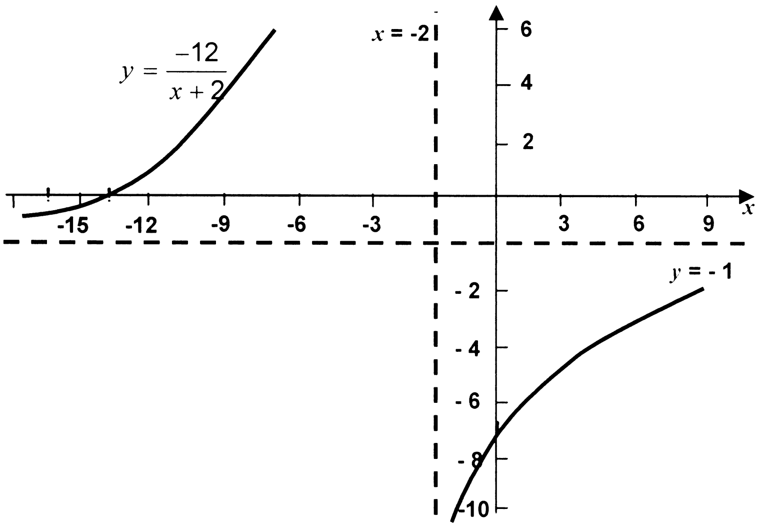
y int. (0 ; 0)

$y = \frac{4}{x}$ is reflected across the y–axis, translated right 1 unit and up 2 units.

2.



3.



$y = \frac{-12}{x+2}$

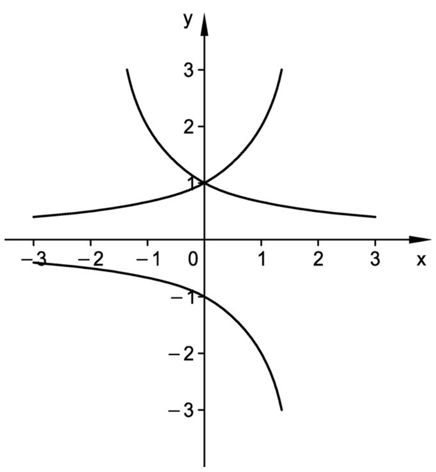
x int. $(-14 ; 0)$

y int. $(0 ; -7)$

Lesson 35

Activity 1

a)



Lesson 36

Activity 1

1. (a) $y - a(x - 2)^2 + 1$ pt $(0 ; 7)$

$7 = a(4) + 1$

$6 = 4a$

$a = \frac{3}{2}$ $y = \frac{3}{2}(x - 2)^2 + 1$

(b) $y = a(x + 2)^2 + 8$ pt $(0 ; -8)$

$-8 = 4a + 8$

$4a = -16$

$a = -4$

$y = -4(x + 2)^2 + 8$

(c) $y = a(x + 1)^2 - 3$ pt $(1 ; 9)$

$9 - 4a - 3$

$4a = 12$

$a = 3$ $y = 3(x + 1)^2 - 3$

(d) $y = a(x + 2)^2 + 2$ pt $(0 ; 0)$

$0 = 4a + 2$

$a = -\frac{1}{2} y = -\frac{1}{2}(x + 2)^2 + 2$

2. (a) $y = a(x - 1)^2 - 1$ pt $(3 ; -5)$

$-5 = a(4) - 1$

$4a = -4$

$a = -1$

$y = -(x - 1)^2 - 1$

(b) $y = -(x + 3)^2 + 3$

Activity 2

1. $y = \frac{k}{x + 1} + 2$ pt $(-2 ; 4)$

$4 = -k + 2$

$k = -2$

$y = \frac{-2}{x + 1} + 2$

2. $y = \frac{k}{x - 2} + 1$ pt $(6 ; 0)$

$0 = \frac{k}{4} + 1$

$k = -4$

$y = \frac{-4}{x - 2}$ y int $(0 ; 3)$

3. $y = \frac{k}{x - 3} - 1$ pt $(4 ; 3)$

$3 = k - 1$

$k = 4$

$y = \frac{4}{x - 3} - 1$ x int $(7 ; 0)$

y int $\left(0 ; -\frac{7}{3}\right)$

4. $y = \frac{k}{x + 2} + 4$ pt (2 ; 3_

$3 = \frac{k}{4} + 4$

$k = -4$

$y = \frac{-4}{x + 2} + 4$ x int $(-1 ; 0)$

y int $(0 ; 2)$

Activity 3

1. pt $\left(0 ; \frac{1}{4}\right)$

$\frac{1}{4} = 2 - p$

$2 - 2 = 2 - p$

$\therefore p = 2$

$y = 2x - 2$

2. $y = 2^{x + p} + 3$ pt $(1 ; 4)$

$4 = 21 + p + 3$

$2(0) = 21 + p$

$p = -1$

$y = 2x - 1 + 3$

y int $\left(0 ; 3\frac{1}{2}\right)$

3. $y = 3^{x+p} + 1$ pt $(1 ; 10)$

$10 = 31 + p + 1$

$32 = 31 + p$

$p = 1$

$y = 3x + 1 + 1$

y int. $(0 ; 4)$

4. pt. $(-1 ; 8)$ is on the graph so

$8 = a.b-1$

$\therefore\ 8 = \frac{a}{b}$

pt $(0 ; 2)$ is on the graph so

$2 = a.b0$

$\therefore\ a = 2$

$\therefore\ 8 = \frac{2}{b}$

$\therefore\ b = \frac{1}{4}$

$y = 2.\left(\frac{1}{4}\right)^{x}$

Lesson 37

Activity 1

1. Shape

$\frac{y}{2} = x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 - \frac{49}{16} - 2$

$y = 2\left(x - \frac{7}{4}\right)^2 - 8\frac{1}{8}$

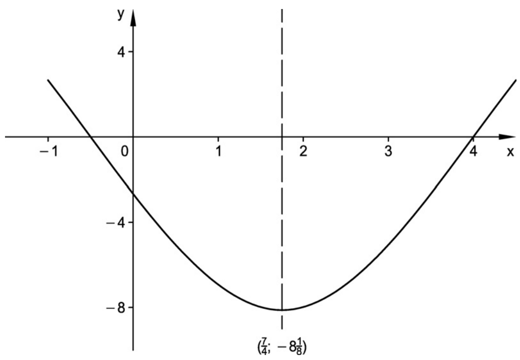
Turning points $\left(\frac{7}{4} ; -8\frac{1}{8}\right)$

y-intercepts $(0 ; -4)$

x-intercepts $0 = 2x^2 - 7x - 4$

$0 = (2x + 1)(x - 4)$

$\left(-\frac{1}{2} ; 0\right)$ $(4 ; 0)$



2. Shape

$-y = x^2 - 8x + (-4)^2 - 16 + 12$

$y = -(x - 4)^2 + 4$

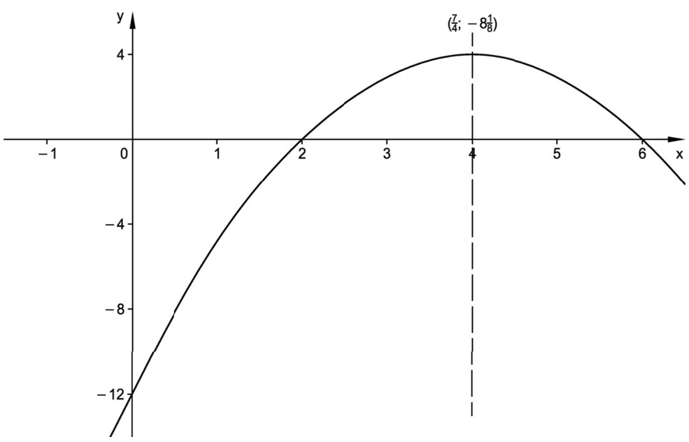
Turning points $(4 ; 4)$

y-intercepts $(0 ; -12)$

x-intercepts $0 = 2x^2 - 8x + 12$

$0 = (x - 6)(x - 2)$

$(6 ; 0)$ $(2 ; 0)$



3. Shape $\cap$

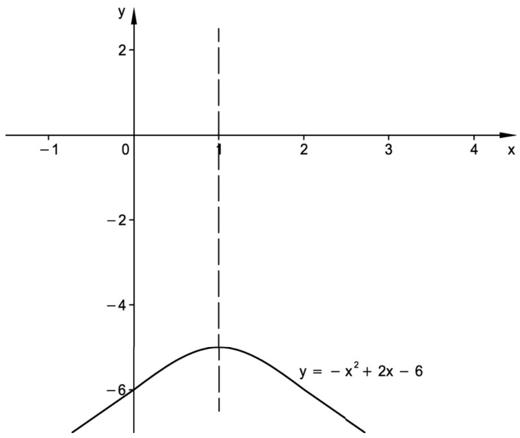
$-y = x^2 - 2x + (-1)^2 - 1 + 6$

$y = -(x - 1)^2 - 5$

Turning points $(1 ; -5)$

y-intercepts $(0 ; -6)$

no x-intercepts



Activity 2

1. (NB: If you prefer drawing the parabola by completing the square – do so.)

a) $y = -x^2 + 3x - 2$ Shape $\cap$

Axis of symmetry $x = \frac{-b}{2a}$

$x = \frac{-3}{-2} = \frac{3}{2}$

Maximum value $y = -\left(\frac{9}{4}\right) + \frac{9}{2} - \frac{2}{1}$

$y = \frac{-9 + 18 - 8}{4} = \frac{1}{4}$

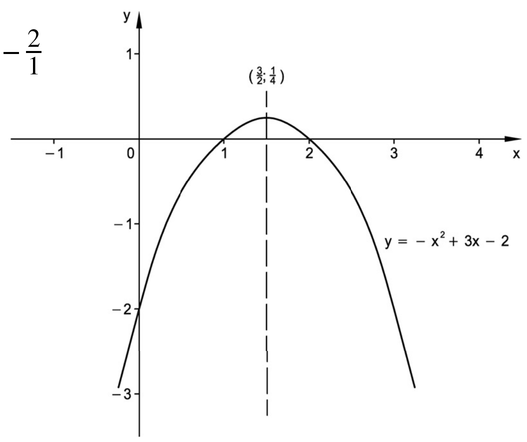
Turning points $\left(\frac{3}{2} ; \frac{1}{4}\right)$

y-intercepts $(0 ; -2)$

x-intercepts $0 = x^2 - 3x + 2$

$0 = (x - 2)(x - 1)$

$(2 ; 0)$ $(1 ; 0)$



b) $y = 2x^2 - 5x - 3$ Shape $\cup$

Axis of symmetry $x = \frac{5}{4}$

Minimum value $y = 2\left(\frac{25}{16}\right) - \left(\frac{5}{4}\right) - \frac{3}{1}$

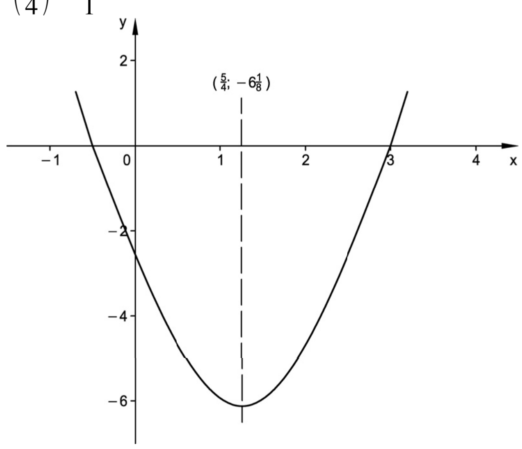
$y = \frac{50 - 100 - 48}{16}$

$= -\frac{98}{16}$

Turning points $\left(\frac{5}{4} ; -6\frac{1}{8}\right)$

y-intercepts $(0 ; -3)$

x-intercepts $\left(0 ; -\frac{1}{2}\right)$ $(3 ; 0)$



c) $y = -2x^2 + 5x$

$\frac{y}{-2} = x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 - \frac{25}{16}$

$y = -2\left(x - \frac{5}{4}\right)^2 + \frac{25}{8}$

Turning points $\left(\frac{5}{4} ; -3\frac{1}{8}\right)$ Shape

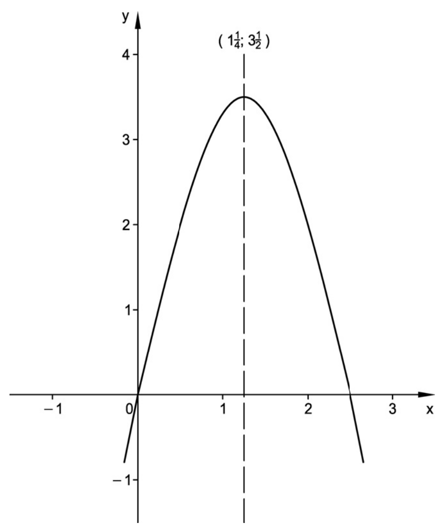
y-intercepts (0 ; 0)

x-intercepts $2x^2 - 5x = 0$

$x(2x - 5) = 0$

$x = 0$ or $x = \frac{5}{2}$

(0 ; 0) $\left(2\frac{1}{2} ; 0\right)$



2. a) $y = a(x - 3)^2 - 1$

$2 = a(9) - 1$

$3 = 9a$

$\therefore\ a = \frac{1}{3}$

$y = \frac{1}{3}(x - 3)^2 - 1$

b) $y = a(x + 1)^2 + 8$

$7 = a(1)^2 + 8$

$a = -1$

$y = -(x + 2)^2 + 8$

c) $y = a(x + 2)^2 + 18$

$0 = a(3)^2 + 18$

$-18 = 9a$

$\therefore\ a = -2$

$y = -2(x - 2)^2 + 18$

3. a) $-x^2 - 6x + 7 = 0$

$x^2 + 6x - 7 = 0$

$(x + 7)(x - 1) = 0$

A(–7 ; 0)

b(1 ; 0) C(0 ; 7)

$y = -x^2 - 6x + 7$

$\frac{y}{-1} = x^2 + 6x + (5)^2 - 9 - 7$

$y = -(x + 3)^2 + 16$

D(–3 ; 16)

b) $-x^2 - 6x + k = 0$ h*a*s no x-intercepts if the original graph is moved 16 units or more down. k is the y–intercept. It was 7 so it moves 16 down, i.e. $k < -9$

c) $-x^2 - 6x + k = 0$ has two negative roots if $-9 \leq k < 0$

4. a) $y = 2p^2x^2 - 2px + 1$

$\frac{y}{2p^2} = x^2 - \frac{1}{p}x + \frac{1}{2p^2}$

$\frac{y}{2p^2} = x^2 - \frac{1}{p}x + \left(\frac{1}{2p}\right)^2 - \frac{1}{42p^2} + \frac{1}{2p^2}$

$\frac{y}{2p^2} = \left(x - \frac{1}{2p}\right)^2 + \frac{2 - 1}{4p^2}$

$y = 2p^2\left(x - \frac{1}{2p}\right)^2 + \frac{1}{2p^2}$

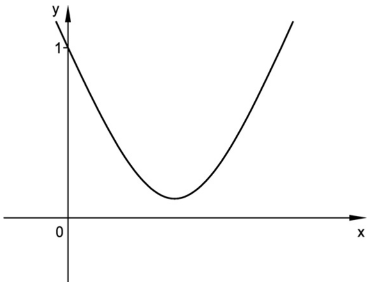
$2p^2 = \left(x - \frac{1}{2p}\right)^2 \geq 0$

$\therefore\ 2p^2\left(x - \frac{1}{2p}\right) + \frac{1}{2p^2} > 0$

Shape $\cup$ Turning points $\left(\frac{1}{2p}\,;\,\frac{1}{2p^2}\right)$

Always positive

$\therefore$ no x-intercepts



b) $\left(\frac{1}{2p}\,;\,\frac{1}{2p^2}\right)$

Lesson 38

1. a) $2x^2 - 4x - 16 = 0$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

A(–2; 0) B(4; 0) C(0; –16) D(1; –18) E(2; –16)

b) $m_{AE} = \frac{16}{-4} = -4$

$y = -4(x + 2)$

$y = -4x - 8 \quad m = -4 \qquad k = -8$

c) PQ = top – bottom

$= 2x^2 - 4x - 16 + 4x + 8$

$= 2x^2 - 8$

Substitute $x = -3$

PQ = 10 units

d) $KL = -4x - 8 - 2x^2 + 4x + 16$

$= -2x^2 - 8$

Maximum is 8 at $x = 0$

2. a) $y = a(x - 1)^2 + 1$

Substitute (0; –3)

$-3 = a(1) + 1$

$a = -4$

$y = -4(x - 1)^2 + 1$ or $y = -4x^2 + 8x - 3$

at C $y = 0$ $\quad 0 = 4x^2 - 8x + 3$

$0 = (2x - 1)(2x - 3)$

$x = \frac{1}{2}$ or $x = \frac{3}{2}$

C$\left(\frac{3}{2}; 0\right)$ T(0; –3)

$m = \frac{-3}{-\frac{3}{2}} = 2 \quad y = 2x - 3$

$f(x) = -4x^2 + 8x - 3$

$g(x) = 2x - 3$

b) $\frac{1}{2} < x < \frac{3}{2}$

c) $0 < x < \frac{3}{2}$

d) Subst. $x = \frac{1}{2}$ into $y = +2x - 3$

$= 1 - 3$

$= |-2|$

AD = 2 units

e) $PQ = -4x^2 + 8x - 3 - 2x + 3$

$= -4x^2 + 6x$

$\frac{PQ}{-4} = x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \frac{9}{16}$

$PQ = -4\left(x - \frac{3}{4}\right)^2 + \frac{9}{4}$ maximum $2\frac{1}{4}$

3. Shape $\cup$ ax of sym $= \frac{2}{2} = 1$

minimum value $y = -4$ turning point $(1; -4)$

y int. $(0; -3)$

$x^2 - 2x - 3 = 0$

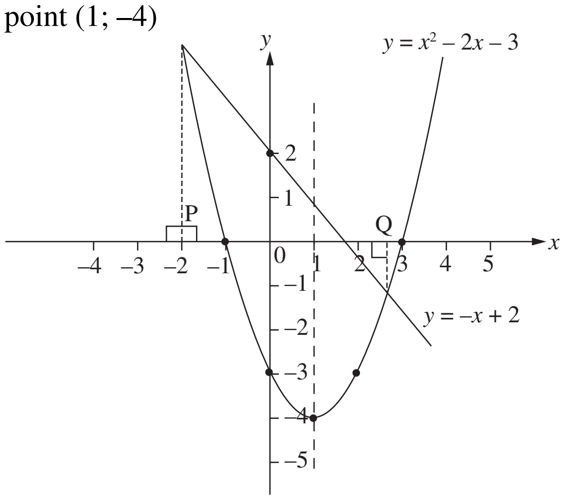
$(x - 3)(x + 1) = 0$

$(3; 0)(-1; 0)$

$x^2 - x - 5 = 0$

$x^2 - 2x - 3 = -x + 2$

draw $y = -x + 2$



4. a) $x = 2 \quad y = -1$

b) A(0; 1) B(–2; 0)

c) D(–2; 5) A(0; 1)

$m = \frac{4}{-2} = -2 \qquad y = -2x + 1$

$m = -2 \qquad k = 1$

d) $-2x + 1 = \frac{-4}{x - 2} - 1$

$-2x + 2 = \frac{4}{x - 2}$

$-2x(2x - 2) + 2(x - 2) = -4$

$-2x^2 + 4x + 2x - 4 = -4$

$0 = 2x^2 - 6x$

$0 = x^2 - 3x$

$0 = x(x - 3)$

C (3; –5)

e) D(–2; 5) C(3; –5)

$DC = \sqrt{25 + 100} = 5\sqrt{5}$

f) $y = -(x - 2) - 1$

$y = -x + 1$

5. a) $y = k^x$ $\quad 2 = \frac{a}{-1}$

$2 = k^{-1}$ $\quad a = -2$

$k = \frac{1}{2}$

$f(x) = \left(\frac{1}{2}\right)^x$ $\quad g(x) = \frac{-2}{x}$

b) A(0; 1) c) $y = -\left(\frac{1}{2}\right)^x$

d) $y = 2^x$ e) $y = \frac{2}{x}$

f) $y = \frac{2}{x}$ g) $h(x) = \frac{-2}{(x + 1)} - 2$

h) $y = -(x + 1) - 2$

$y = -x - 3$

6. a) $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

B(–1; 0) $\quad \therefore$ AB = 3

b) $PQ = x + 4 + x^2 - 2x - 3$

$7 = x^2 - x + 1$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

$x = 3$ or $x = -2$

OR = 2 units

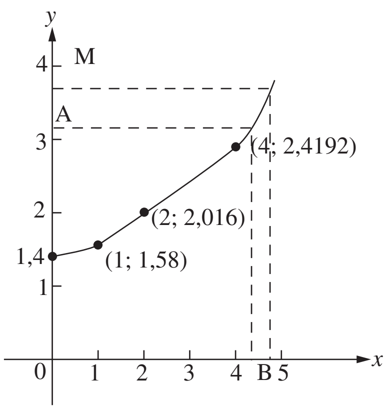
c) $CD = x^2 - x + 1$

$CD = x^2 - x + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1$

$CD = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

minimum value $\frac{3}{4}$ at $x = \frac{1}{2}$

7. a)



b) At A

c) 3,18 kg

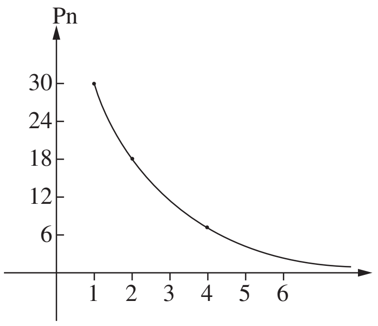
d) At B ≅ 4,9 months

e) $M = 1{,}8(1{,}2)^x$

8. a) Sequence

$36; 18; 9; \frac{9}{2}$

b)



c) The triangle will get very very small.

9. a) $\frac{4}{x-4} = -2$

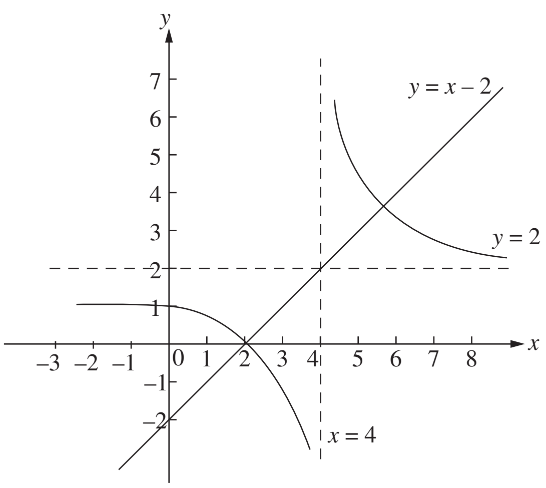
$4 = -2x + 8$

$2x = 4 \qquad \therefore x = 2$

b) $p = -1 + 2$

$p = 1$

c)



d) $y \in \Re - \{2\}$

e) $y = (x - 4) + 2$

$y = x - 2$

10. a) 2 in 6 hours (1 in 12 hours)

4 in 3 hours

b) $\frac{12}{0{,}4} = 5$

c)

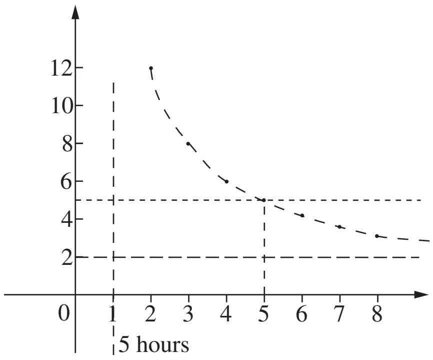
x	1	2	4	5
y (hrs)	12	6	3	2,4

d) $y = \frac{12}{x}$

e) $y = \frac{12}{x} + 2$

f) $y = \frac{12}{(x-1)} + 2$

g) Translated up 2 and to the right 1



i) On graph

Lesson 39

1. S = {a; b; c; d; e; f; g; h; i; j}

   (i) $\frac{5}{10} = \frac{1}{2}$ (ii) $\frac{1}{5}$ $n(\text{S}) = 10$

   (iii) $\frac{1}{10}$ (iv) $\frac{7}{10}$

   (v) $\frac{3}{10}$ (vi) $\frac{1}{10}$

2. Let Event A: additional mathematics pupils
   M: mathematics pupils
   W: science pupils

   $n(\text{S}) = 126$

   a) $\frac{4}{126} = \frac{2}{63}$ b) $\frac{3}{31}$ c) $\frac{7}{63}$ d) $\frac{25}{63}$ e) $\frac{122}{126} = \frac{61}{63}$

3. a) 0,1 b) 0,2 c) 0,3 d) 0,9

4. a) 0,45 b) 0,15 c) 0,6 d) 0,7 e) 0,7

   f) 0,75 g) 0,3

5. $0{,}48 - x + x + 0{,}5 - x = 0{,}72$

   $0{,}98 - x = 0{,}72$

   $0{,}26 = x$

   $P(A \cap B) = 0{,}26$

6. $\frac{5}{8} - x + x + \frac{5}{6} - x = \frac{15}{16}$

   $30 - 48x + 48x + 40 - 48x = 45$

   $70 - 48x = 45$

$25 = 48x$

$x = \frac{25}{48}$

a) $\frac{25}{48}$ b) $\frac{5}{6} - \frac{25}{48} = \frac{40 - 25}{48} = \frac{15}{48}$ c) $\frac{1}{16}$

7. $\frac{16}{52} = \frac{4}{13}$

8. $0{,}7 - x + x + 0{,}3 - x = 0{,}8$

   $1 - x = 0{,}8$

   $0{,}2 = x$

   a) 0,2 b) 0,6 c) 0,2

9. Draw and put the intersection in first.

   $0{,}5 + 0{,}3 - x + 0{,}4 = 1$

   $-x = 1 - 1{,}2$

   $x = 0{,}2$

   (i) 0,2

   (ii) 0,3

10. $19 + 17 - x + 14 + 7 + 11 + 8 + 12 = 80$

    $-x = -8$

    $x = 8$

    a) 12

    b) 8

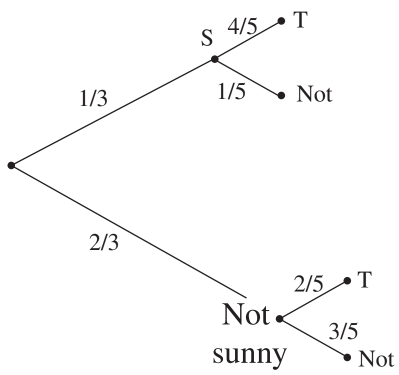
    c) $\frac{40}{80} = \frac{1}{2}$

Lesson 40

1. P(Jenny plays tennis)

   $= \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{5}\right)$

   $= \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$

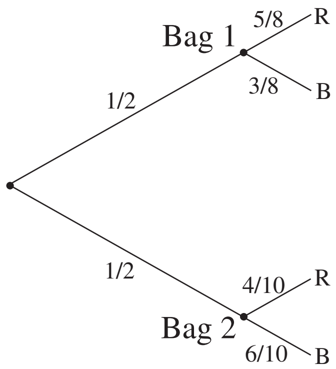


2. P(Blue)

$= \left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$

$= \frac{3}{16} + \frac{3}{10}$

$= \frac{15 + 24}{80} = \frac{39}{80}$



3. a) P(accident)

$= (0{,}25)(0{,}02) + (0{,}75)(0{,}004)$

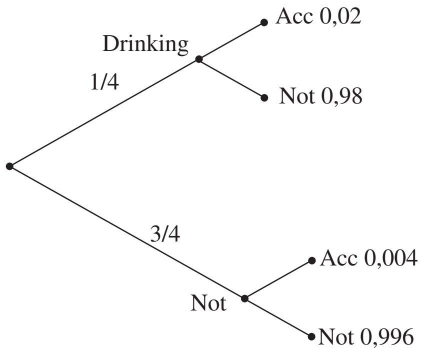
$= 0{,}008$

b) Not really on the syllabus but interesting.

$\text{P(drunk)} = \frac{\text{P(Drinking and Accident)}}{\text{P(accident)}}$

$= \frac{(0{,}25)(0{,}02)}{0{,}008}$

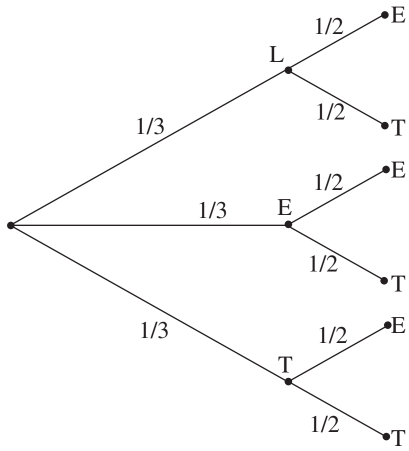
$= 0{,}625$



4. a) $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$

b) $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)$

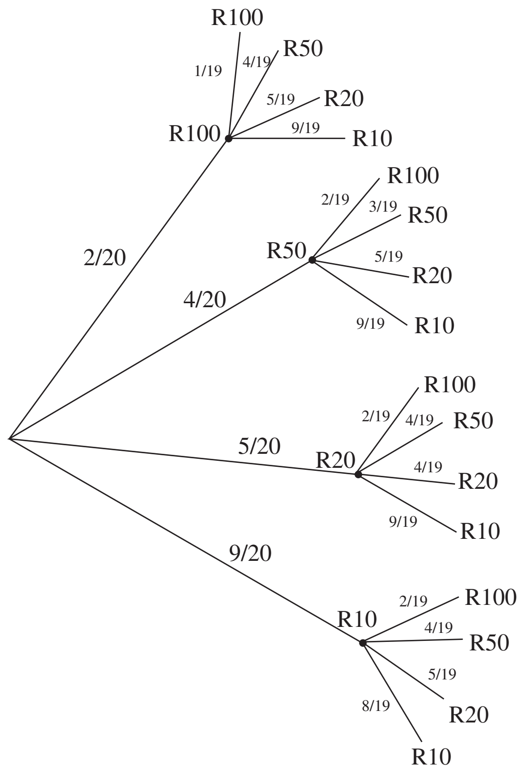
$= \frac{2}{3}$



5. a) $\frac{4}{20} \times \frac{5}{19}$

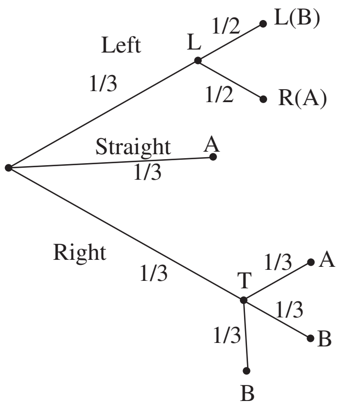
$= \frac{1}{19}$

b) $\left(\frac{4}{20}\right)\left(\frac{5}{19}\right) + \left(\frac{4}{20}\right)\left(\frac{9}{19}\right) + \left(\frac{5}{20}\right)\left(\frac{4}{19}\right) + \left(\frac{5}{20}\right)\left(\frac{9}{19}\right) + \left(\frac{9}{20}\right)\left(\frac{4}{19}\right) +$
$\left(\frac{9}{20}\right)\left(\frac{5}{19}\right) + \left(\frac{9}{20}\right)\left(\frac{8}{19}\right)$



6. $P(A) = \frac{1}{6} + \frac{1}{3} + \frac{1}{9} = \frac{11}{18}$

$P(B) = \frac{1}{6} + \frac{1}{9} + \frac{1}{9} = \frac{7}{18}$



Lesson 41

Activity 1

1. (a) $A = 4\,000(1 + 0{,}14 \times 5) = R6\,800$ (b) $A = 4\,000(1 + 0{,}14)^5 = R7\,701{,}66$

2. (a) $13\,000 = P(1 + 0{,}12 \times 6)$

$\therefore\ 13000 = P(1{,}72)$

$\therefore \frac{13\,000}{(1{,}72)} = P$

$\therefore\ P = R7\,558{,}14$

(b) $13\,000 = P(1 + 0{,}12)^6$

$\therefore 13\,000 = P(1{,}12)^6$

$\therefore\ \frac{13\,000}{(1{,}12)^6} = P$

$\therefore\ P = R6\,586{,}20$

3. $A = 70\,000(1 + 0{,}09 \times 3)\ .\ (1 + 0{,}08)^2$

$\therefore\ A = R103\,692{,}96$

Activity 2

(a) R60 000 (b) R20 000 (c) 7 years (d) 8 years (e) zero rands

Activity 3

1. (a) $A = 250\,000(1 - 0{,}12 \times 5) = R100\,000$ (b) $A = 250\,000(1 - 0{,}12)5 = R131\,932{,}98$

2. (a) $A = P(1 - in)$

$\therefore\ 3\,200 = P(1 - 0{,}12 \times 7)$

$\therefore\ 3\,200 = P(0{,}16)$

$\therefore\ \frac{3\,200}{0{,}16} = P$

$\therefore\ P = R20\,000$

b) $A = P(1 - i)n$

$\therefore\ 3\,200 = P(1 - 0{,}12)^7$

$\therefore\ 3\,200 = P(0{,}88)7$

$\therefore\ \frac{3\,200}{(0{,}88)^7} = P$

$\therefore\ P = R7\,830{,}17$

3. (a) $19\,000 = 140\,000(1 - 10i)$

$19\,000 = 140\,000(1 - 10i)$

$\therefore\ 19\,000 = 140\,000 - 1\,400\,000i$

$\therefore\ 1\,400\,000i = 121\,000$

$\therefore\ i = 0{,}08642857143$

$\therefore\ r = 8{,}6\%$

(b) $19\,000 = 140\,000(1 - i)10$

$\therefore\ \frac{19\,000}{140\,000} = (1 - i)10$

$\therefore \left(\frac{19\,000}{140\,000}\right)^{\frac{1}{10}} = 1 - i$

$\therefore\ i = 1 - \left(\frac{19\,000}{140\,000}\right)^{\frac{1}{10}}$

$\therefore\ i = 0{,}1810402522$

$\therefore\ r = 18{,}1\%$

4. $56\,000 = P(1 - 0{,}14)^5$

$\therefore\ 56\,000 = P(0{,}86)^5$

$\therefore\ \frac{56\,000}{(0{,}86)^5} = Pw$

$\therefore\ P = R110\,040{,}78$

5. $A = 400\,000(1 - 0{,}16 \times 6)$

$\therefore\ A = 16\,000$

6. $A = 750\,000\,(1 - 0{,}13)6$

$\therefore\ A = R325\,219{,}65$

Lesson 42

Activity 1

1. (a) $A = 20000(1 + 0{,}15)^6 = R46261{,}22$

(b) $A = 20000\left(1 + \frac{0{,}15}{2}\right)^{12} = R47635{,}59$

(c) $A = 20000\left(1 + \frac{0{,}15}{4}\right)^{24} = R48388{,}76$

(d) $A = 20000\left(1 + \frac{0{,}15}{12}\right)^{72} = R48918{,}41$

(e) $A = 20000\left(1 + \frac{0{,}15}{365}\right)^{2190} = R49182{,}97$

2. Annual compounding:

$A = 60000(1 + 0{,}12)^{15} = R328412{,}95$

Monthly compounding:

$A = 60000\left(1 + \frac{0{,}12}{12}\right)^{180} = R359748{,}12$

The monthly compounding would give you R31334,17 more.

3. $A = 30000\left(1 + \frac{0{,}16}{4}\right)^{16} \cdot \left(1 + \frac{0{,}15}{2}\right)^{10} = R115808{,}20$

4. A. $A = x\left(1 + \frac{0{,}14}{12}\right)^{12} = x(1{,}149342029)$

B. $A = x\left(1 + \frac{0,16}{4}\right)^4 = x(1,16985856)$

Option B is the better option.

5. Present value at T_0

$1200000 = P(1 + \frac{0,18}{12})^{72}$

$\therefore 1200000 = P(1,015)^{72}$

$\therefore \frac{1200000}{(1,015)^{72}} = P$

$\therefore P = R410796$

Activity 2

1. (a)
$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$
$$\therefore 1 + i_{eff} = \left(1 + \frac{0,14}{2}\right)^2$$
$$\therefore i_{eff} = (1,07)^2 - 1$$
$$\therefore i_{eff} = 0,1449$$
$$\therefore r_{eff} = 14,5\%$$

(b)
$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$
$$\therefore 1 + i_{eff} = \left(1 + \frac{0,16}{4}\right)^4$$
$$\therefore i_{eff} = (1,04)^4 - 1$$
$$\therefore i_{eff} = 0,16985856$$
$$\therefore r_{eff} = 17\%$$

(c)
$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$
$$\therefore 1 + i_{eff} = \left(1 + \frac{0,12}{12}\right)^{12}$$
$$\therefore i_{eff} = (1,01)^{12} - 1$$
$$\therefore i_{eff} = 0,1268250301$$
$$\therefore r_{eff} = 12,7\%$$

(d)
$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$
$$\therefore 1 + i_{eff} = \left(1 + \frac{0,10}{365}\right)^{365}$$
$$\therefore i_{eff} = \left(1 + \frac{0,10}{365}\right)^{365} - 1$$
$$\therefore i_{eff} = 0,1051557816$$
$$\therefore r_{eff} = 10,5\%$$

2. (a)
$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$
$$\therefore 1 + 0,132 = \left(1 + \frac{i_{nom}}{4}\right)^4$$
$$\therefore 1,132 = \left(1 + \frac{i_{nom}}{4}\right)^4$$
$$\therefore (1,132)^{\frac{1}{4}} = 1 + \frac{i_{nom}}{4}$$
$$\therefore 4\left[(1,132)^{\frac{1}{4}} - 1\right] = i_{nom}$$
$$\therefore 0,1259275539 = i_{nom}$$
$$\therefore r_{nom} = 12.6\%$$
$$\therefore r = 12.6\% \text{ per ann c.q.}$$

(b)
$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$
$$\therefore 1 + 0,145 = \left(1 + \frac{i_{nom}}{2}\right)^2$$
$$\therefore 1,145 = \left(1 + \frac{i_{nom}}{2}\right)^2$$
$$\therefore (1,145)^{\frac{1}{2}} = 1 + \frac{i_{nom}}{2}$$
$$\therefore 2\left[(1,145)^{\frac{1}{2}} - 1\right] = i_{nom}$$
$$\therefore 0,1400934559 = i_{nom}$$
$$\therefore r_{nom} = 14\%$$
$$\therefore r = 14\% \text{ pa csa}$$

$$\therefore 1 + 0,105 = \left(1 + \frac{i_{nom}}{12}\right)^{12}$$
$$\therefore 1,105 = \left(1 + \frac{i_{nom}}{12}\right)^{12}$$
$$\therefore (1,105)^{\frac{1}{2}} = 1 + \frac{i_{nom}}{12}$$
$$\therefore 12\left[(1,105)^{\frac{1}{12}}\right] - 1 = i_{nom}$$
$$\therefore 0,1002618682 = i_{nom}$$
$$\therefore r_{nom} = 10\%$$

(c) $1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$

3. (a) $A = 24000\left(1 + \frac{0,16}{4}\right)^{48} = R157692,68$

(b) $1 + i_{eff} = \left(1 + \frac{0,16}{4}\right)^4$

$\therefore\ i_{eff} = (1,04)^4 - 1$

$\therefore\ i_{eff} = 0,16985856$

$\therefore\ r_{eff} = 17\%$

(c) $A = 24000(1 + 0,16985856)^{12} = R157692,68$

$\therefore\ 1 + 0,0446975070792 = \left(1 + \frac{i_{nom}}{12}\right)^{12}$

$\therefore\ 1 + 0,0446975070792 = \left(1 + \frac{i_{nom}}{12}\right)^{12}$

$\therefore\ (1 + 0,0446975070792)^{\frac{1}{12}} = \left(1 + \frac{i_{nom}}{12}\right)$^

$\therefore\ 12\left[(1 + 0,0446975070792)^{\frac{1}{12}} - 1\right] = i_{nom}$

$\therefore\ i_{nom} = 0,04380714361$

4. (a) $650000 = 500000 = (1 + i_{eff})^6$

(b) $1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$

$\therefore\ \frac{650000}{500000} = (1 + i_{eff})^6$

$\therefore\ \left(\frac{65}{50}\right)^{\frac{1}{6}} = 1 + i_{eff}$

$\therefore\ \left(\frac{65}{50}\right)^{\frac{1}{6}} - 1 = i_{eff}$

$\therefore\ i_{eff} = 0,04469750792$

$\therefore\ r_{eff} = 4,5\%$

Lesson 43

Activity 1

1. $A = 5000\left(1 + \frac{0,07}{4}\right)^{16}\left(1 + \frac{0,08}{2}\right)^{12} = R10566,25$

2. $A = 2000\left(1 + \frac{0,18}{12}\right)^{48}\left(1 + \frac{0,24}{2}\right)^{6} = R8066,93$

3. $A = 3000\left(1 + \frac{0,13}{12}\right)^{60}\cdot\left(1 + \frac{0,14}{12}\right)^{36}\cdot\left(1 + \frac{0,12}{12}\right)^{24} = R11039,65$

4. (a) $1 + \mathrm{i}_{eff} = \left(1 + \frac{0,08}{12}\right)^{12}$ $\quad$ $1 + \mathrm{i}_{eff} = \left(1 + \frac{0,10}{2}\right)^{2}$

$\therefore \mathrm{i}_{eff} = \left(1 + \frac{0,08}{12}\right)^{12} - 1$ $\quad$ $\therefore \mathrm{i}_{eff} = \left(1 + \frac{0,10}{2}\right)^{2} - 1$

$\therefore\ \mathrm{i}_{eff} = 0,08299950681$ $\quad$ $\therefore \mathrm{i}_{eff} = 0,1025$

(b) $A = 6000(1,08299950681)^7\cdot(1,1025)^5 = R17078,20$

Activity 2

1. $P = 13000\left(1 + \frac{0,12}{12}\right)^{-36}(1 + 0,09)^{-4} = R6436,77$

2. $P= 10000000\left(1 + \frac{0,20}{4}\right)^{-8}\left(1+ \frac{0,15}{12}\right)^{-72} = R2767217,60$

3. $P50000\left(1 + \frac{0,11}{12}\right)^{-24}\left(1 + \frac{0,14}{4}\right)^{-4} = R35002,50$

4. (a) $P = 100000\left(1 + \frac{0,13}{2}\right)^{-4}\left(1+ \frac{0,14}{12}\right)^{-24} = R58844,11$

   (b) $1 + i_{eff} = \left(1 + \frac{0,14}{12}\right)^{12}$ $\quad 1 + i_{eff} = \left(1 + \frac{0,13}{2}\right)^{2}$

   $\therefore\ i_{eff} = 1 + = \left(1 + \frac{0,14}{12}\right)^{12} - 1$ $\quad \therefore\ i_{eff} = 1 + = \left(1 + \frac{0,13}{2}\right)^{2} - 1$

   $\therefore\ i_{eff} = 0,1493420292$ $\quad \therefore\ i_{eff} = 0,134225$

   (c) $P = 100000(1 + 0,134225)^{-2}(1 + 0,1493420292)^{-2}$
   $= R\ R58844,11$

   The same amount for P as in 4(a) is obtained.

Lesson 44

Activity 1

1. $A = 3500\left(1 + \frac{0,08}{12}\right)^{24}\left(1 + \frac{0,10}{2}\right)^{8} + 4000\left(1 + \frac{0,10}{2}\right)^{6} = R11425,50$

2. $A = 5000\left(1 + \frac{0,13}{2}\right)^{8}\left(1 + \frac{0,14}{4}\right)^{12} + 4000\left(1 + \frac{0,13}{2}\right)^{2}\left(1 + \frac{0,14}{4}\right)^{12} + 6000$

   $\therefore\ A = R25359,63$

3. $A = 2000\left(1+ \frac{0,09}{12}\right)^{12}\left(1+ \frac{0,08}{12}\right)^{24} + 4000\left(1 + \frac{0,08}{12}\right)^{24} + 8000\left(1 + \frac{0,08}{12}\right)^{12}$

   $\therefore\ A = R15921,37$

4. $x\left(1+ \frac{0,18}{12}\right)^{60} + 2x\left(1 + \frac{0.18}{12}\right)^{36} + 3x = 60000$

   $\therefore\ x = \left[\left(1 + \frac{0,18}{12}\right)^{60} + 2\left(1 + \frac{0,18}{12}\right)^{36} + 3\right] = 6000$

   $\therefore\ x = 6000/\left(1 + \frac{0,18}{12}\right)^{60} + 2\left(1 + \frac{0,18}{12}\right)^{36} + 3$

   $\therefore\ x = R677,09$

Activity 2

1. $A = 15000\left(1 + \frac{0,13}{2}\right)^{12} - 3000\left(1 + \frac{0,13}{2}\right)^{4}$

   $\therefore\ A = R28077,04$

2. $A = 5000\left(1 + \frac{0,08}{12}\right)^{36}\left(1 + \frac{0,09}{4}\right)^{16} + 6000\left(1 + \frac{0,08}{12}\right)^{12}\left(1 + \frac{0,09}{4}\right)^{16} - 3000\left(1 + \frac{0,09}{4}\right)^{8}$

$\therefore$ A = R14759,27

3. $A = 4300\left(1 + \frac{0,13}{12}\right)^{36}\left(1 + \frac{0,14}{4}\right)^{16} + 7000\left(1 + \frac{0,14}{4}\right)^{16} - 2000\left(1 + \frac{0,09}{4}\right)^{8} + 1000$

$\therefore$ A = R21737,74

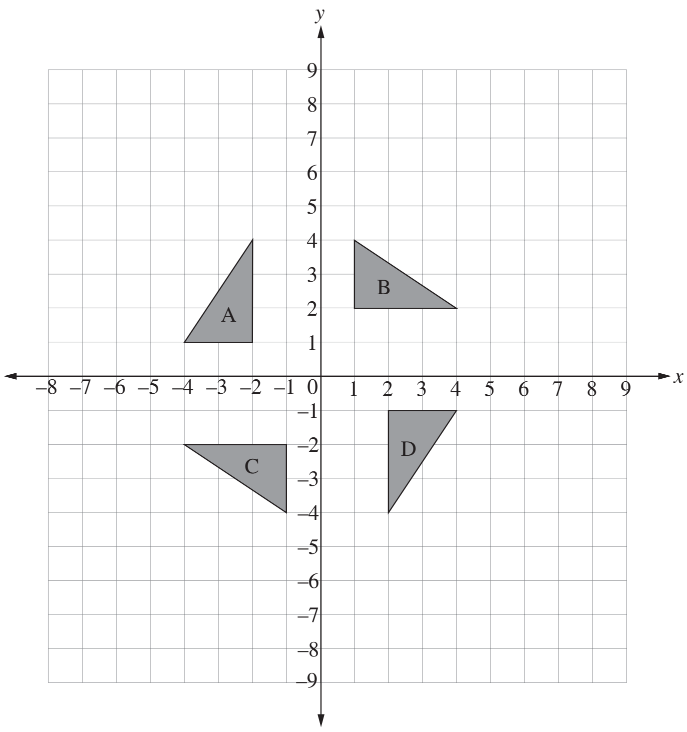
Activity 3

1. $P = 13000\left(1 + \frac{0,12}{4}\right)^{-12}\left(1 + \frac{0,09}{12}\right)^{-48} = R6369,92$
2. $P = 2300\left(1 + \frac{0,10}{2}\right)^{-8} + 4200\left(1 + \frac{0,12}{12}\right)^{-24}\left(1 + \frac{0,10}{2}\right)^{-10} = R3587,42$
3. $P = 6000 + 8000\left(1 + \frac{0,18}{12}\right)^{-24} + 900\left(1 + \frac{0,19}{12}\right)^{-24}\left(1 + \frac{0,18}{12}\right)^{-24} = R10319,06$
4. $A = 30000 + 20000\left(1 + \frac{0,18}{12}\right)^{-24} + 100000(1 + 0,32)^{-5}\left(1 + \frac{0,18}{12}\right)^{-36}$

$\therefore$ A = R58590,88

Lesson 46

Activity 1

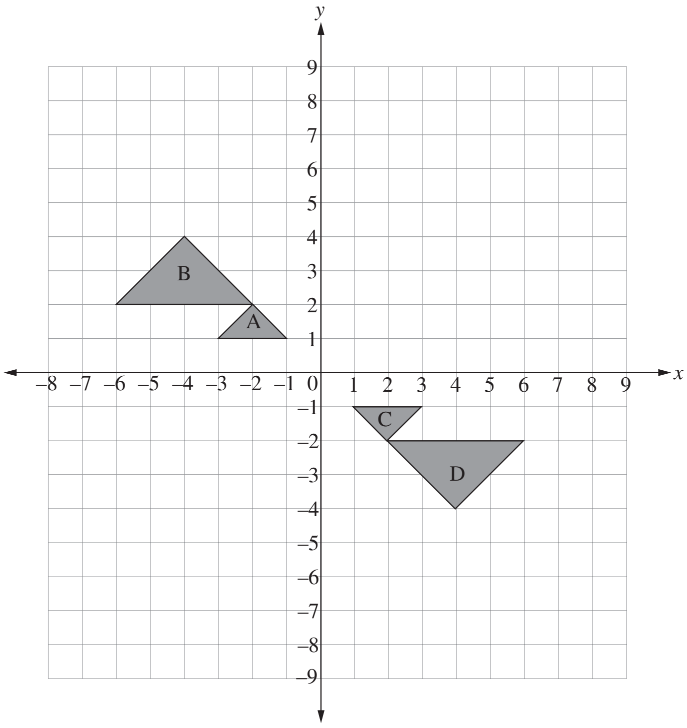


B: Rotation of 90° clockwise

C: Rotation of 90° anti-clockwise

D: Rotation of 180°

Activity 2

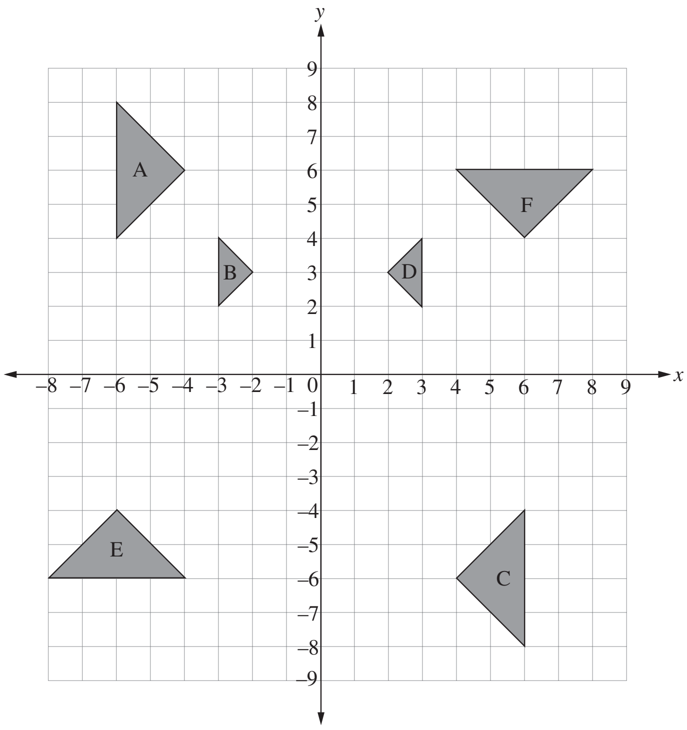


B: Enlargement by a scale factor of 2

C: Rotation of 180°

D: Combined rotation of 180° and enlargement

Activity 3



Lesson 47

Activity 1

1. A. The position of $Q_2 = \frac{1}{2}(19 + 1) = 10$.

The **Median** of the data is 16 (the 10th value).

The position of $Q_1 = \frac{1}{4}(19 + 1) = 5$

The **Lower Quartile** of the data is 9 (the 5th value). It is a part of the data set.

The position of $Q_3 = \frac{3}{4}(19 + 1) = 15$

The **Upper Quartile** of the data is 21 (the 15th value). It is part of the data set.

A	2	3	5	7	9	10	11	13	15	16	16	17	18	19	21	22	23	25	32	

B. The position of $Q_2 = \frac{1}{2}(17 + 1) = 9$th position.

The **Median** of the data is the 9th value:

$Q_2 = 15$

The position of $Q_1 = \frac{1}{4}(17 + 1) = 4{,}5$th position

The **Lower Quartile** of the data is the average between the 4th and 5th value:

$Q_1 = \frac{7 + 9}{2} = 8$ (not part of the data set)

The position of $Q_3 = \frac{3}{4}(17 + 1) = 13{,}5$th position

The **Upper Quartile** of the data is the average between the 13th and 14th value:

$Q_1 = \frac{18 + 19}{2} = 18{,}5$ (not part of the data set)

B	2	3	5	7	8	9	10	11	13	15	16	16	17	18	18,5	19	21	22	23

C. The position of $Q_2 = \frac{1}{2}(20 + 1) = 10{,}5$ (average of the 10th and 11th)

The **Median** of the data is $\frac{16 + 16}{2} = 16$ (not in data set)

Since n is even and since $\frac{n}{2} = \frac{20}{2} = 10$ which is even, the lower and upper quartiles will not be values in the data set.

The position of $Q_1 = \frac{1}{4}(20 + 1) = 5{,}25$ (average of the 5th and 6th value).

The **Lower Quartile** of the data is $\frac{9 + 10}{2} = 9{,}5$

The position of $Q_3 = \frac{3}{4}(20 + 1) = 15{,}75$ (average of the 15th and 16th).

The **Upper Quartile** of the data is $\frac{21 + 22}{2} = 21{,}5$

C	2	3	5	7	9	9,5	10	11	13	15	16	16	16	17	18	19	21	21,5	22	23	25	32	34

D. The position of $Q_2 = \frac{1}{2}(18 + 1) = 9{,}5$th position (average of the 9th and 10th value).

The **Median** of the data is $\frac{15 + 16}{2} = 15{,}5$

Since n is even and since $\frac{n}{2} = \frac{18}{2} = 9$ which is odd, the lower and upper quartiles will be values in the data set.

The position of $Q_1 = \frac{1}{4}(18 + 1) = 4{,}75$ (5th value)

The **Lower Quartile** of the data is 9.

The position of $Q_3 = \frac{3}{4}(18 + 1) = 14{,}25$ (14th value).

The **Upper Quartile** of the data is

D	2	3	5	7	9	10	11	13	15	15,5	16	16	17	18	19	21	22	23	25

2. First arrange in ascending order.

10A	11	12	14	15	15	16	16	16	16	16	16	17	17	19	22	22	23	24	26	–
10B	8	9	10	13	14	14	14	14	16	17	18	19	20	20	23	24	27	28	29	30
10C	2	5	7	12	12	12	14	14	14	14	14	14	14	20	21	21	24	26	–	–

(a) Mean for 10A: $\frac{333}{19} = 17,5$

Mean for 10B: $\frac{367}{20} = 18,4$

Mean for 10C: $\frac{260}{18} = 14,4$

(b) Mode for 10A: 16

Mode for 10B: 14

Mode for 10C: 14

(c) Median for 10A: 16

Median for 10B: 17,5

Median for 10C: 14

(d) Range for 10A: $26 - 11 = 15$

Range for 10B: $30 - 8 = 22$

Range for 10C: $26 - 2 = 24$

(e) Lower quartile for 10A: $Q_1 = 15$

Lower quartile for 10B: $Q_1 = 14$

Lower quartile for 10C: $Q_1 = 12$

(f) Upper quartile for 10A: $Q_3 = 22$

Upper quartile for 10B: $Q_3 = 23,5$

Upper quartile for 10C: $Q_3 = 20,5$

(g) Interquartile range for 10A: $Q_3 - Q_1 = 22 - 15 = 7$

Interquartile range for 10B: $Q_3 - Q_1 = 23,5 - 14 = 9,5$

Interquartile range for 10C: $Q_3 - Q_1 = 20,5 - 12 = 8,5$

(h) Semi-interquartile range for 10A: $\frac{Q_3 - Q_1}{2} = \frac{22 - 15}{2} = 3,5$

Semi-interquartile range for 10B: $\frac{Q_3 - Q_1}{2} = \frac{23,5 - 14}{2} = 4,75$

Semi-interquartile range for 10C: $\frac{Q_3 - Q_1}{2} = \frac{20,5 - 12}{2} = 4,25$

3.

1	9	10	4	7	4	4	10
7	3	9	3	8	9	3	7
7	3	9	4	5	8	6	6
1	3	10	2	2	7	8	7
7	2	7	6	2	8	7	6

Marks (10)	Tally	Frequency (f)	Mark *x* f
1	|	2	2
2	||||	4	8
3	|||| ~~||||~~	5	15
4	||||	4	16
5	|	1	5
6	||||	4	24
7	~~||||~~ ||||	9	63
8	||||	4	32
9	||||	4	36
10	|||	3	30
		Total: 40	Total: 231

Mean $= \frac{231}{40} = 5{,}8$

Activity 2

1. (a) A: $\overline{x} = \frac{79}{13} = 6{,}1$ B: $\overline{x} = \frac{87}{12} = 7{,}3$

C: $\overline{x} = \frac{52}{11} = 4{,}7$ D: $\overline{x} = \frac{72}{12} = 6$

(b) Driver A: Driver B: Driver C: Driver D:

Min: 1 Min: 1 Min: 1 Min: 2

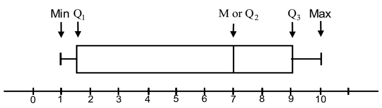
$Q_1 = 1{,}5$ $Q_1 = 7$ $Q_1 = 2$ $Q_1 = 3$

$Q_2 = 7$ $Q_2 = 8$ $Q_2 = 4$ $Q_2 = 6$

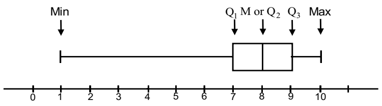
$Q_3 = 9$ $Q_3 = 9$ $Q_3 = 8$ $Q_3 = 9$

Max: 10 Max: 10 Max: 10 Max: 10

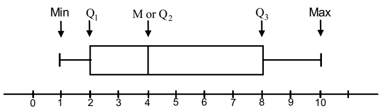
(c) Driver A:



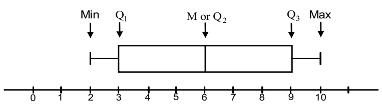
Driver B:



Driver C:



Driver D:



(d) Driver A: Skewed to the left of the median.

Median is to the right of the mean. Data is negatively skewed.

Driver B: Symmetrical with respect to the median.

Median is to the right of the mean. Data is negatively skewed.

Driver C: Skewed to the right of the median.

Median is to the left of the mean. Data is positively skewed.

Driver D: Symmetrical with respect to the median.

Data is symmetrical about the mean.

(e) Driver B performed the best because 50% of his points are 8 or above.

Driver A had the second best performance because 50% of his points are 7 or above.

Driver D had the third best performance because 50% of his points are 6 or above.

Driver A did the worst because 50% of his points are 4 or above.

Lesson 48

Activity 1

Complete this exercise in this workbook.

The following table contains the number of learners who obtained certain marks on a class test out of 30.

Marks	20	21	22	23	24	25	26	27	28	29
No of learners	3	3	4	5	7	10	13	5	4	2

(a) Draw a cumulative frequency table for this data.

Marks	Frequency	Cumulative Frequency
20	3	3
21	3	6
22	4	10
23	5	15
24	7	22
25	10	32
26	13	45
27	5	50
28	4	54
29	2	56
Total	56	

(b) On the set of axes provided on the next page, draw a cumulative frequency graph (ogive curve) for this data.

(c) Determine the lower quartile.

Position of $Q_1 = \frac{1}{4}(57) = 14{,}25$

$\frac{56}{2} = 28$ which is even (Q_1 and Q_3 are not part of data).

$Q1 = \frac{23 + 23}{2} = 23$

(d) Determine the median.

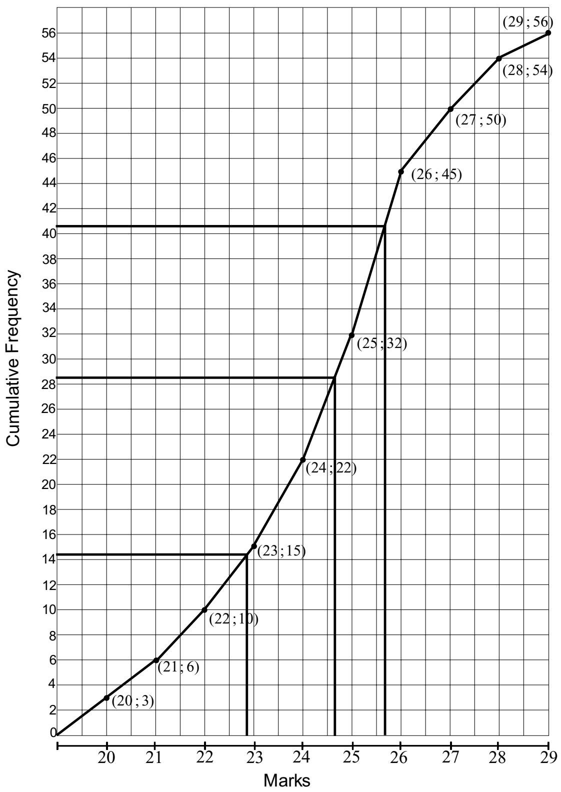
Position of $Q_2 = \frac{1}{2}(57) = 28{,}5\text{th}$

$$Q_2 = \frac{25 + 25}{2} = 25$$

(e) Determine the upper quartile.

Position of $Q_3 = \frac{3}{4}(57) = 42{,}25\text{th}$

$$Q_3 = \frac{26 + 26}{2} = 26$$



Activity 2

1. The following table (grouped frequency distribution) shows the mark obtained by 220 learners in a Science examination.

Percentage	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	2	6	11	22	39	59	45	20	11	5

(a) Complete the cumulative frequency table for this data.

Marks	**Frequency**	**Cumulative Frequency**
1-10	2	2
11-20	6	8
21-30	11	19
31-40	22	41
41-50	39	80
51-60	59	139
61-70	45	184
71-80	20	204

81-90	11	215
91-100	5	220
Total	220	

(b) On the set of axes provided on the next page, draw a cumulative frequency graph (ogive curve) for this data.

(c) Determine the lower quartile.

Position of $Q_1 = \frac{1}{4}(221) = 55{,}25\text{th}$

$Q_1 \approx 44$ (read off from graph)

(d) Determine the median.

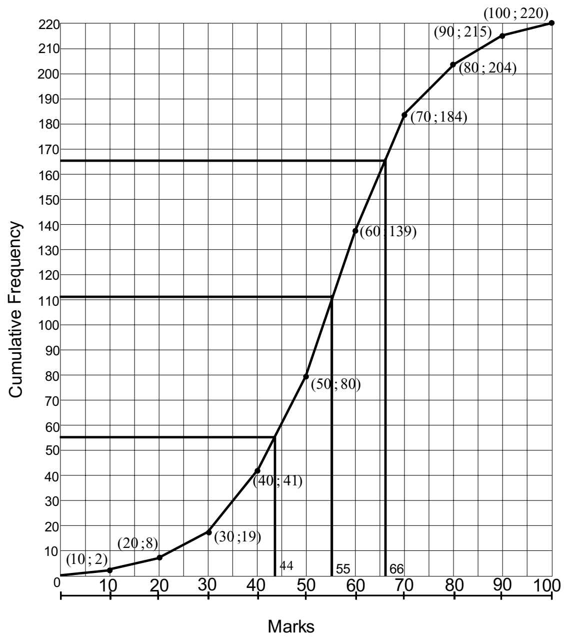
Position of $Q_2 = \frac{1}{2}(221) = 110{,}5\text{th}$

$Q_2 \approx 55$ (read off from graph)

(e) Determine the upper quartile.

Position of $Q_3 = \frac{3}{4}(221) = 165{,}75\text{th}$

$Q_3 \approx 55$ (read off from graph)



2. The table represents the percentage of income spent on recreation by 50 people.

(a) Complete the cumulative frequency table for this data.

Percentage	**Frequency**	**Cumulative Frequency**
$12 < p \leq 18$	8	8
$18 < p \leq 24$	20	28
$24 < p \leq 30$	12	40
$30 < p \leq 36$	8	48
$36 < p \leq 42$	2	50
Total	50	

(b) On the set of axes provided on the next page, draw a cumulative frequency graph (ogive curve) for this data.

(c) Determine the lower quartile.

$\frac{50}{2} = 25$ which is off (Q_1 and Q_3 are part of data)

Position of $Q_1 = \frac{1}{4}(51) = 12,75 = 13$th position

$Q_1 \approx 19,5$ (read off from graph)

(d) Determine the median.

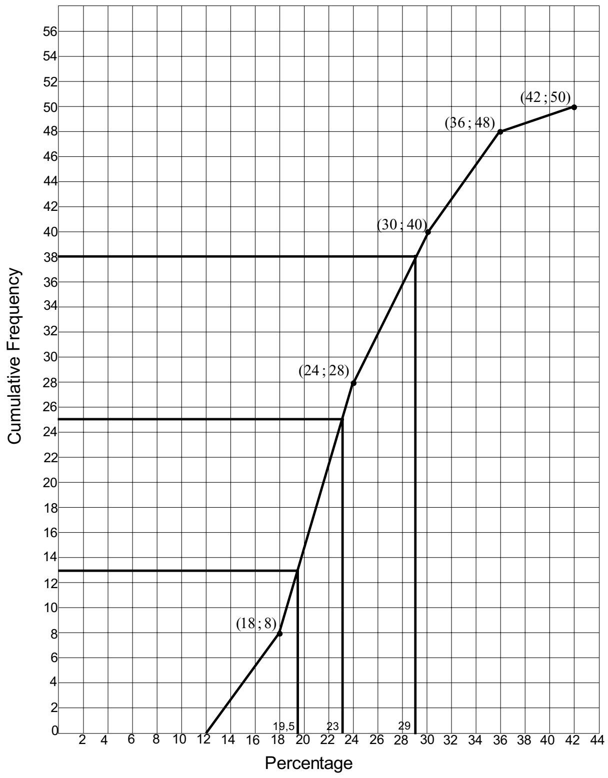
Position of $Q_2 = \frac{1}{2}(51) = 25,5$th

$Q_2 \approx 23$ (read off from graph)

(e) Determine the upper quartile.

Position of $Q_3 = \frac{3}{4}(51) = 38,25 = 38$th

$Q_3 \approx 29$ (read off from graph)



Lesson 49

Activity 1

1. (a) A: $\bar{x} = \frac{79}{13} = 6,1$ B: $\bar{x} = \frac{87}{12} = 7,3$

C: $\bar{x} = \frac{52}{11} = 4,7$ D: $\bar{x} = \frac{72}{12} = 6$

(b) Driver A:

x	$(x - \bar{x})$	$(x - \bar{x})^2$
1	–5,1	26,01
1	–5,1	26,01
1	–5,1	26,01
2	–4,1	16,81
6	–0,1	0,01
6	–0,1	0,01
8	1,9	3,61
8	1,9	3,61
8	1,9	3,61
8	1,9	3,61
10	3,9	15,21
10	3,9	15,21
10	3,9	15,21
		$\sum(x - \bar{x})^2 = 154,93$

variance $(s^2) = \frac{\sum(x - \bar{x})^2}{n} = \frac{154,93}{13} = 11,91769231$

standard deviation (s) = $\sqrt{\frac{\sum(x - \bar{x})^2}{n}} = 3,45$

Driver B:

x	$(x - \bar{x})$	$(x - \bar{x})^2$
1	–6,3	39,69
2	–6,3	28,09
6	–1,3	1,69
8	0,7	0,49
8	0,7	0,49
8	0,7	0,49
8	0,7	0,49
8	0,7	0,49
8	0,7	0,49
10	2,7	7,29
10	2,7	7,29
10	2,7	7,29
		$\sum(x - \bar{x})^2 = 94,28$

variance $(s^2) = \frac{\sum(x - \bar{x})^2}{n} = \frac{94,28}{12} = 7,856666667$

standard deviation (s) = $\sqrt{\frac{\sum(x - \bar{x})^2}{n}} = 2,8$

Driver C:

x	$(x - \bar{x})$	$(x - \bar{x})^2$
1	–3,7	13,69
1	–3,7	13,69
2	–2,7	7,29
2	–2,7	7,29
4	–0,7	0,49
4	–0,7	0,49
6	1,3	1,69
6	1,3	1,69

8	3,3	10,89
8	3,3	10,89
10	5,3	28,09
		$\sum(x-\bar{x})^2 = 96,19$

variance (s²) $= \frac{\sum(x-\bar{x})^2}{n} = \frac{96,19}{11} = 8,744545455$

standard deviation (s) $= \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = 2,96$

Driver D:

x	$(x-\bar{x})$	$(x-\bar{x})^2$
2	–4	16
2	–4	16
2	–4	16
4	–2	4
4	–2	4
6	0	0
6	0	0
8	2	4
8	2	4
10	4	16
10	4	16
10	4	16
		$\sum(x-\bar{x})^2 = 112$

variance (s²) $= \frac{\sum(x-\bar{x})^2}{n} = \frac{112}{12} = 9,\dot{3}$

standard deviation (s) $= \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = 3,1$

(c) Driver B's performance is clustered closely around its mean of 7,3 in comparison with the other drivers. The mean is the highest for B. Therefore, B performed the best.

2.

Marks x	**Freq f**	$f \times x$	$(x-\bar{x})$	$(x-\bar{x})^2$	$f \times (x-\bar{x})^2$
20	3	60	–4,8	23,04	69,12
21	3	63	–3,8	14,44	43,32
22	4	88	–2,8	7,84	31,36
23	5	115	–1,8	3,24	16,2
24	7	168	–0,8	0,64	4,48
25	10	250	0,2	0,04	0,4
26	13	338	1,2	1,44	18,72
27	5	135	2,2	4,84	24,2
28	4	112	3,2	10,24	40,96
29	2	58	4,2	17,64	35,28
Total	**56**	$\bar{x} = \frac{1387}{56} = 24,8$			284,04

variance (s²) $= \frac{\sum f(x-\bar{x})^2}{n} = \frac{284}{56} = 5,072142857$

standard deviation (s) $= \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = 2,25$

Activity 2

1. The table represents the percentage of income spent on recreation by 50 people.

   (a) Complete the following table for this data.

Percentage	f	Midp (m)	$f \times m$	$m - \bar{x}$	$(m - \bar{x})^2$	$f \times (m - \bar{x})^2$
$12<p \leq 18$	8	15	120	–9,1	82,81	662,48
$18<p \leq 24$	20	21	420	–3,1	9,61	192,2
$24<p \leq 30$	12	27	324	2,9	8,41	100,92
$30<p \leq 36$	8	33	264	8,9	79,21	633,68
$36<p \leq 42$	2	39	78	14,9	222,01	444,02
Total	50		$\bar{x} = \frac{1206}{50} = 24{,}1$			2033,3

   (b) Calculate the mean for this data

   $\bar{x} = \frac{1206}{50} = 24{,}1$

   (c) Now calculate the standard deviation for this data.

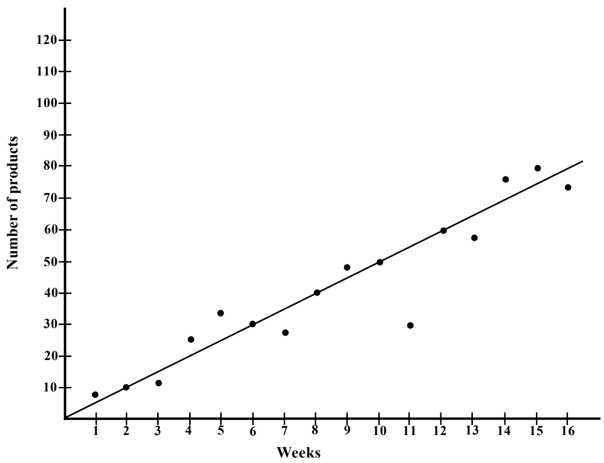
$$\text{variance } (s^2) = \frac{\sum f(x - \bar{x})^2}{n} = \frac{2033{,}3}{50} = 40{,}666$$

$$\text{standard deviation } (s) = \sqrt{\frac{\sum f(xf - \bar{x})^2}{n}} = 6{,}4$$

   (d) Now verify your answer by using a calculator.

Lesson 50

1. A company records the number of products sold per week.



   (a) Draw a line of best fit on the diagram above.

   (b) Determine the equation of your line of best fit.

   (2; 10)(6;30)

   gradient $= \frac{30 - 10}{6 - 2} = \frac{20}{4} = 5$ y-intercept is 0

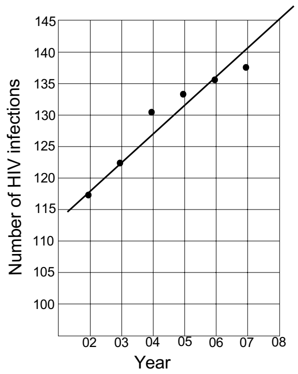
   $\therefore\ y = 5x$ is the equation of the line of best fit

(c) Predict how many products will be sold after 40 weeks.

$y = 5(40) = 200$

An estimated 200 products might be sold.

2. (a)



(b) There is a line of best fit for this data.

(c) There would probably be an estimated 145 infections in 2008.

TIPS FOR TEACHERS

Lesson 1

- It is important to first revise factorisation of algebraic expressions with your learners before tackling this lesson.
- Make sure that your learners do not cancel incorrectly.

  $\frac{\cancel{x^2} + x}{\cancel{x^2} - 1}$ is canceling incorrectly. Learners must first factorise and then cancel:

  $\frac{x\cancel{(x + 1)}}{\cancel{(x + 1)}(x - 1)} = \frac{x}{x - 1}$
- Encourage your learners to make use of the very important results discussed in this lesson.
- Emphasises the difference between simplifying an expression and solving an equation. Be careful of incorrect use of equal sign with equations. Learners tend to misuse the equal sign as follows:

  $2 = \frac{5}{x - 1}$

  $\quad = 2(x - 1) = 5$

  $\quad = 2x - 2 = 5$

  Rather encourage them to use the therefore sign when solving equations:

  $\therefore 2 = \frac{5}{x - 1}$

  $\therefore 2(x - 1) = 5$

  $\therefore 2x - 2 = 5$
- Make sure that learners understand the concept of taking out a negative when factorising an expression:

  $-3x + 12 = -3(x - 4)$ but not $-3(x + 4)$

  $-3x - 12 = -3(x + 4)$ but not $-3(x - 4)$

Lesson 2

When we complete the square we write a quadratic expression in the form $a(x - p)^2 + q$

- If $a < 0$ then q is the maximum value of the expression at $x = p$.
- If $a < 0$ and q is less than 0, the expression will always be negative.
- If $a > 0$ then q is the minimum value of the expression when $x = p$.
- If $a > 0$ and q is greater than 0, then the expression will always be positive.

Lesson 3

- Examples should include simple concepts to test understanding and not complicated algebraic manipulation.
- Encourage learners to memorize powers of 2 up to 2^6, powers of 3 to 3^4 and powers of 5 to 5^3.
- Stress to them never to divide across + and – signs but to factorise.
- The investigations can be used for portfolio exercises and marked by the rubric at the end of the lesson.

Rubric for investigations

Presentation of work	Messy and unable to read. ①	Very little logical sequence. ②	Neat and tidy but not enough. ③	Untidy but correct. ④	Tidy work and correct. ⑤
<u>Investigation1</u> Mathematics	Did not understand what mathematics to use. ①	Only able to perform mathematical operations with help and mistakes were made. ②	Correct mathematics with help from the teacher. ③	Well investigated with few errors. ④	Everything correct. ⑤
Constructions	Did not know what to do. ①	Needed help with constructions and not completely accurate. ②	Constructions correct but help was needed. ③	Work done on their own but some concepts not included. ④	Everything correct. ⑤
<u>Investigation2</u> Mathematics	Did not see the sequence as the mathematics is incorrect. ①	Needed help with Pythagoras and only one length investigated (no nth term). ②	Investigated different lengths but could not get the nth term. ③	Investigated different lengths and obtained the nth term for each sequence. ④	Everything correct and a general term found for any length. ⑤

TOTAL: 20

Lesson 4

- Worksheet 2 can be used for portfolio exercises.
- Make sure that the learners make the quadratic equation equal to zero.
- It is a good idea to note the restrictions when they write down the LCD.
- Yet they should still be encouraged to check solutions.
- Learners should know the difference between an identity and an equation.
- When they need to prove a conjecture they must make the left-hand side equal to the right-hand side.

Lesson 5

It is a good idea to put an asterisk next to the surd before the learners square so that they remember to check.

Drawing the graph by testing points helps the learners to understand the restrictions.

Lesson 6

Stress to the learners that solving equations by completing the square is like a theorem and should only be used when they are told to.

They could learn this algorithm in step form where dividing by the coefficient if x squared is golden rule number one. Always make them aware of when there are solutions and when they actually could have factorised (rational solutions).

Lesson 7

Alert the learners to read the question carefully and decide how they must produce the answer (decimal place or surd form or, in fact, simple surd form).

The nature of roots is no longer in the curriculum but I do think it is still important for the learners to analyse the quadratic formula and decide how they will solve the equation.

Lesson 8

Encourage learners to look at the second line and if it is too complicated, create a k substitute. They need to remember to substitute back. They must check $b^2 - 4ac$ to see if they can factorise.

Lesson 10

- Allow learners to go straight to the number line and put the critical values on the number line. Otherwise they often make mistakes with the inequality signs.
- Inequalities with rational expressions (variables in the denominator) are no longer on the syllabus, so if you do them, it will only be for extension.

Lesson 11

- The routine-type questions such as speed, distance and time are important to help learners formulate equations, but I think it is important that they learn to cope with non-routine questions.
- Encourage the learners to draw pictures and make tables.
- Remember the best way to make equations is :
  Big minus small equals the difference.

Lesson 12–13

- It is important to teach this section slowly because the learners battle with the signs.
- Encourage the learners to use their calculators to check answers.
- Let the learners experiment with the calculator.

Lesson 14

- When the learners have to prove the left-hand side is the same as the right-hand side, it is a good idea to draw a line down the middle so that they do not treat the identity as an equation.
- The only time they need to change 1 into $\sin^2 \alpha + \cos^2 \alpha$ is to create factors.
- Encourage the learners to work inside the brackets.

Lesson 15

- Make sure the learners use brackets after they have reduced the ratios otherwise they get confused with addition and multiplication.
- When they are doing multiplication encourage them to look for factors to cancel.
- Even though they can always find the trig ratios of angles on their calculators, it is good training and enhances understanding to reduce.

Lesson 16–18

One of the reasons why we no longer teach sec θ cosec θ and cot θ is because the equations were too complicated, so I suggest keeping the factors as simple as possible.

Learners can work in pairs and make up equations for their partners to solve.

Lesson 19–20

Let the learners work together in this section.

They learn to use their calculators and at the same time handle the sequences.

Lesson 21

- This whole section is best done as portfolio work.
- Learners could be asked to do projects and research on the Golden Ratio, Pascal's triangle and various sequences of numbers.

Lesson 22

- This section of work should certainly be done before volume and area because many of these concepts are needed
- The learners do need to know theorems.
- Emphasis will be placed on actual angles and not on variables.
- Encourage learners to reproduce diagrams and sometimes draw their own diagrams.

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- The learners do need to know theorems.
- Emphasis will be placed on actual angles and not variables.
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Lesson 24

- Emphasis must be placed on real life type problems.

- This section can be used for portfolio type work – learners can be asked to use trigonometry to find heights and angles of objects in their daily lives such as trees, rugby posts, etc.

Lesson 25

- The best way to explain the more difficult three dimensional shapes is to make the learners construct models.
- Group work is fun to use in this section.

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Lesson 27

Inclination and straight line equations are meant to be done in Grade 11, but it is necessary to revise the formulae done in Grade 10. This Lesson basically covers the Grade 10 work.

Lesson 28

- Learners work nicely in pairs in this section.
- Always encourage the learners to draw diagrams.
- Analytical Geometry is about making equations.
- Learners should not copy out the formula from the formula sheet, they must substitute directly into the formula.

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Lesson 31

- This section will be examined in Paper 3 and is not compulsory.
- However, it is important that as many children as possible write this paper.
- When doing geometry it is a good idea to let the learners work together in groups.
- Geometry is a discipline, so the setting out of the work is very important.

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- Geometry is a discipline so the setting out of the work is very important.

Lesson 35

- Graphs are a huge part of the curriculum so we need to do them slowly and carefully.
- It is a good idea to put the learners in pairs and let them give each other graphs to draw.

Lesson 36

- This is a good section to do in pairs or in groups.
- The emphasis must be on translating graphs.

Lesson 37

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Lesson 39

- This is an optional section of the curriculum at present but will become an important section in the future. It is important to start teaching this section.
- Most countries in the world study probability and it is a very important section needed to prepare learners for tertiary education.

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Lesson 41

- Learners need to understand the difference between simple and compound interest as well as the difference between linear and reducing balance depreciation.
- It is extremely important for learners to know the formulae for simple and compound interest as well as the two formulae for depreciation, even though these formulae may be provided in a formula sheet in the final examinations.
- It is vital for teachers to link the concepts of linear and exponential functions to these formulae.
- Learners must be able to efficiently use their calculators when doing the calculations. The use of the CASIO fx-82ES calculator is recommended.
- The use of time lines (see first example In this lesson) is highly recommended, especially when learners are required to answer more complicated questions (as will be seen in the next Lessons).
- In questions requiring the calculation of the interest rate, learners must ensure that they determine the interest rate as a percentage. Leaving the answer as a decimal is not sufficient.

Lesson 42

- Learners should understand that the nominal rate (taking different compounding periods into account), and the effective annual rate will yield the same accumulated amount at the end of the investment period. You could demonstrate this by using the idea of two people investing the same amount of money for one year and receiving the same accumulated amount at the end of that year. Person A invests money and the interest is calculated, say, every month. Person B invests money and interest is calculated at the end of the year. The yearly effective rate is clearly greater than the nominal rate, but the accumulated amounts will be the same for both people. However, the quoted rate without compounding cannot yield the same accumulated amount.

For example, suppose that James and John each invest R2 000 for one year at 18% per annum.

- The interest rate for James is 18% per annum compounded monthly (nominal).
- The interest rate for John is the effective annual rate of 19,195618171%.

They will both receive the same accumulated amount at the end of the year, even though the interest rates differ.

James: $A = 2000\left(1 + \frac{0,18}{12}\right)^{12} = R2391,24$

(Interest is calculated at the end of each month)

John: $A = 2000(1 + 0,195618171)^{1} = R2391,24$

(Interest is calculated at the end of the year)

You use the formula $1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^{n}$ to convert 18% per annum compounded monthly to the annual effective rate of 19,6%.

- When converting from the nominal to the effective annual rate using the formula, the calculation is always done for one year, even though the question might be asking for accumulated amounts over more than one year (see example 3).

Lesson 43

- Learners should be encouraged to use the methods as illustrated in the summary example rather than to remain with the long methods. This will save a lot of time and produce more accurate answers.
- The effective annual rates should not be rounded off when using them in calculations.
- Learners should not round off interest rate calculations such as $\frac{0,13}{12} = 0,0108\dot{3}$.

It is more effective to leave the interest rate as $\frac{0,13}{12}$ in the calculation.

Lesson 44

- For more manageable calculations, it is highly recommended that educators encourage learners to use the recommended methods in this lesson.

Lesson 45

- It is essential for learners to investigate and discover the transformation rules for themselves.
- Learners must be able to draw images under different transformation rules as well as be able to describe the rule when a given figure is transformed into an image.
- The use of the $(x; y) \rightarrow \ldots$ notation is essential for learners to become familiar with.

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- The use of the $(x; y) \rightarrow \ldots$ notation is essential for learners to become familiar with.

Lesson 47

- It is extremely important for learners to understand how to locate the quartiles in data with odd and even numbers of values.
- Quartiles can form part of the data set or stand outside the data as averages between two data values.
- Comparing Box and Whisker plots are important for exam purposes.

Lesson 48

It is extremely important for learners to understand the difference when working with ungrouped and grouped data. With ungrouped data, the intervals on the horizontal axis will contain values which are the same. To illustrate this, consider the first example. The values in the interval $24 < x \leq 25$ are all 25.

With grouped data, the values in these horizontal axis intervals are not the same. Refer to the second example and you will find that marks in the interval $9 < x \leq 19$ vary from 10 to 19. This means that for ungrouped data, you read off the value in the interval exactly, whereas with grouped data, you have to read off a value approximately by estimating. There are sophisticated methods of estimating the quartiles more accurately. However, these methods are beyond the scope of the Grade 11 syllabus.

Lesson 49

Learners need to be encouraged to master the calculator technique of calculating the standard deviation for given data. The long method is cumbersome and should be discouraged with large sets of data. However, the principle can be examined but with a very small set of data values.

Lesson 50

In Grade 11, learners are expected to intuitively draw a line of best fit. Therefore any feasible line of best fit will be accepted. In Grade 12, they use regression to find the true line of best fit. Correlation is explored in more depth in the Grade 12 year.